condition that two substitution groups are the same, according to this definition, is that one of them can be transformed into the other by means of some substitution. The most general definition of what is meant by the same group can be obtained from this oldest definition by adding the condition that the two substitution groups in question shall be regular and therefore implies that a necessary and sufficient condition that two groups are the same is that a simple isomorphism can be established between their substitutions or operators. The other extreme definition implies that two groups are the same if and only if they involve the same substitutions or operators irrespective of the arrangement of these substitutions or operators within the group.

Just as the two different definitions of the term circle have persisted for at least two and a half millenniums, so these three different definitions of the expression *the same group* may reasonably be expected to persist. When new terms relating to these definitions are introduced, it should be made clear which definition is to be understood. Sometimes this can readily be inferred from the context but at other times it may be necessary to state the matter explicitly in order to avoid confusion. At any rate, the beginner should be informed early as regards the significance of these various definitions and that contradictions are likely to arise therefrom.

Another fundamental question which may give rise to contradictory results on the part of the student of group theory has recently presented itself to me in very definite form by the assertion in C. J. Keyser's *Mathematical Philosophy*, 1922, page 204, that when all the positive and negative rational integers, including zero, are combined according to subtraction they do *not* constitute a group. While I am not inclined to call this assertion incorrect I would have been inclined to agree also with Professor Keyser if he had stated that they *do* constitute a group. While these assertions are evidently contradictory, it depends entirely on what is understood by the rule of combination of the elements of a group whether the one or the other is correct. If each of the rational integers is understood to represent the operation of subtracting it from something; that is, if the integer n is understood to mean that n is to be subtracted, then these integers, together with zero, evidently do constitute the same group as when they are combined by addition.

On the other hand, if S_1 and S_2 represent any two of these integers and if when S_1 and S_2 are combined it means that the integer represented by S_2 is to be subtracted from that represented by S_1, or vice versa, it is evident that the associative law is not satisfied since $S_1 - S_2 - S_3$ is not generally equal to $S_1 - (S_2 - S_3)$. It should, however, be noted that when S_1, S_2, S_3 represent ordinary substitutions, and if by the product $S_1 \cdot S_2$ we should mean that the substitution S_2 were to be performed on S_1; that is, if $S_1 \cdot S_2$ should be understood to mean $S_2^{-1} S_1 S_2$, then the associative law would also not be satisfied by the ordinary substitutions. By combining two substitutions we do not mean that the second substitution is to be applied on the first but we mean merely that one of these substitutions is to be followed by the other.

It would seem natural to say that the substitutions of a group are combined according to the rule of substitutions. In this case, we would mean thereby that the symbols of our group would represent substitutions and that the rule of combination of these symbols was determined by the properties of these substitutions. In any group in which the elements represent definite operations the law of combination of these elements is determined by the properties of the elements. The principal thing is the properties of the elements. The law of combination of the elements, being deduced from these properties, may be regarded as secondary.

I emphasize these points here because from some of the well-known definitions of an abstract group one might naturally conclude that the law of combination of the elements was of primary importance in group theory and that the properties of the elements themselves played a secondary rôle, while just the opposite seems to me to be the case. In the definition of an abstract group it would seem desirable to state explicitly that the law of combination of the elements is deduced from the properties of the elements. In particular, if we speak of groups whose elements are the rational positive and negative integers together with zero, and if we say that these integers are to be combined according to the rule of subtraction, it would seem to me natural to say that a symbol like 4 was to be construed to mean that the number represented by this symbol was to be subtracted, while the symbol -4 was to be regarded as meaning that the number -4 was to be subtracted, or that 4 was to be added.

If this is done, each of these numbers represents a certain operation which may be applied on something outside of the group. In fact, a group generally looks outside of itself for its main fields of usefulness and the law of combination of its elements is normally construed by this outward look as may be inferred from the fact that this law is commonly denoted by the single operation multiplication. In looking outward it seems to me that we must say that the operations of performing ordinary substitutions, as well as rational integers when combined by subtraction, obey the associative law. In confining our view to the system of elements themselves one may still construe the combination of these elements in such a way that the associative law is satisfied but in this case it may seem more natural to say that the associative law is not satisfied when the elements are combined by subtraction.

The fundamental nature of these remarks may be inferred from the fact that Professor Keyser was writing for the educated laymen and not mainly for the mathematicians. The above remarks therefore show that even those who are not professional mathematicians may come across contradictions in this field if they are so fortunate as to catch the spirit of the group concept, which Professor Keyser pictured for them not only in our commonplace mundane surroundings but also in the movements of the heavenly messengers of good tidings to those who are faithful to the end, as is evidenced by the group of angel flights.

He even raises the question: "Is mind a group?" and states that "it may be that a genius of the so-called universal type—an Aristotle, for example, or a

Leibniz or a Leonardo da Vinci—is one whose mind has the group property." This possible fact may serve to explain some of the extraordinary claims made for group theory which many would be inclined to say could be readily contradicted if they had not been made by men of such eminence. For instance, one of the last public utterances of H. Poincaré was that "the theory of groups is, so to say, entire mathematics, divested of its matter and reduced to a pure form."[1] I presume that those who are most familiar with the varied scientific activities of H. Poincaré would be inclined to call him a genius of the universal type.

At any rate, it is true that while some groups are such omnipresent objects that familiarity has blunted our appreciation of their significance there are others which exhibit themselves only to the most gifted or to those who have made a serious search for them. It is a singular fact that the former did not receive any explicit attention until some of the latter were discovered and their rôle in the theory of algebraic equations became apparent. How different mathematical history would be if Archimedes had emphasized the importance of the group concept in the development of our subject. In particular, the later Greeks would then not have been satisfied with one solution of a quadratic equation but they would have recognized that there is no such thing as an equation in one unknown and of degree $n > 1$, but that the so-called equation in one unknown is really one in n unknowns, n being the degree of the equation. They would have seen that in such an equation one has been led to call different things by the same name x in spite of the fact that one may have desired to do otherwise. Algebra is more resourceful than the inventors thereof.

Group theory as an autonomous subject was initiated by looking below the surface and noting that what is denoted by a single symbol x is really not a single number but a composite of n numbers in disguise. It is the mind not the eye that saw these groups, and ever since then it is the groups which the mind sees which have led to many sweeping remarks relating to the scope of their theory. To those of us whose intellectual insight is less keen some of these remarks may appear to be contradicted by facts just because we see these facts only dimly and fail to recognize some deep connections.

A contradiction of a very well-known elementary theorem on substitution groups is found in the 1900 edition of Pascal's *Repertorium der höheren Mathematik*, volume 1, page 32. It is here stated, as well as in the original Italian edition, that an invariant subgroup of a transitive substitution group does *not* involve all the marks or letters of the group. The following sentence is also evidently inaccurate, since the group in question does not necessarily include all the substitutions but only all the positive substitutions. On the same page it is also incorrectly stated that if all the substitutions of a k times transitive group transform a given group into itself, then this group must be at least $k - 1$ times transitive. It is very well known that the holomorph of the regular abelian group of order 2^n and of type $(1, 1, 1, \cdots)$ is triply transitive. In particular, the holomorph of the non-cyclic regular group of order 4 is the symmetric group of degree

[1] *Acta Mathematica*, vol. 38, 1921, p. 145.

4. It is singular that three such obvious errors should appear on the same page of the standard reference work in question.

On page 203 of Keyser's *Mathematical Philosophy* the term "system having the group property" is defined in practically the same way as the term "group in the most general sense" is defined in volume 2, page 243, of tome 1 of the *Encyclopédie des Sciences Mathématiques*. Hence his statement "that in the older literature of the subject they are called groups, or closed systems" is correct but misleading. On page 576 of volume 1 of the *Encyclopédie* there appears a much more restrictive definition of the term group, and no reference is given from either one of these definitions to the other. These two widely different definitions of this term in such a standard work of reference are a reflection of the fact that the word "group" as a technical mathematical term has been employed by good authorities with widely different meanings. It is only natural that this wide difference in meaning has led to contradictions, but it has not produced chaos in the field of group theory, since one can usually determine from the context what meaning is to be assigned to the term in question.

Perhaps many of us would be inclined to think that such an authoritative work as the *Encyclopédie* should have used its influence to secure uniformity as regards the use of the term "group" with a technical meaning. In so far as uniformity reduces suggestiveness it is an evil, especially in the newer fields of our subject. It is much better to encounter contradictions than sterility. In so far as the use of the same term with widely different meanings suggests analogies and common ground which would otherwise escape the reader's notice it is an advantage. On the other hand, clear thinking demands that the technical terms in use have definite meaning in so far as this meaning relates to the main objects of thought under consideration. A man may work very successfully in the ordinary theory of substitution groups, for instance, without considering all the conditions which an abstract group is supposed to satisfy, or without having grasped the reason why the group of Euclidean geometry does not include the similarity transformations but does include the special homothetic transformations in which the coefficients are plus or minus unity.

The most conspicuous contradictions in the literature of group theory naturally appear in the history of this subject since the wide range of topics with which the mathematical historian is inclined to deal makes it very difficult to avoid misinterpretations. As an instance of this difficulty we may refer to the term *simple group*. This term has been used with at least three different meanings. P. Ruffini used its equivalent (permutazione semplice) in his *Teoria generale delle equazioni*, 1799, page 247, for a cyclic group. H. W. Tanner used the same term for a cyclic prime-power group in volume 20, 1888, of the *Proceedings of the London Mathematical Society*, page 70. This obsolete meaning unfortunately slipped into Dickson's *History of the Theory of Numbers*, volume 1 (1919), page 131. It is somewhat singular that the meaning of this term which is now universally adopted, and was introduced by C. Jordan in 1869, *Mathematische Annalen*, volume 1, page 142, is less in accord with the non-technical meaning

of simple than those noted above, for the modern simple groups are really the most difficult groups.

In a semi-serious manner we may refer here to the fact that one of my colleagues recently directed attention to my questionable use of the term *Lagrange's theorem* for the fundamental theorem that the order of a finite group is divisible by the order of each of its subgroups, since Lagrange did not use this theorem with its general meaning.[1] Such a use was made of it by Galois and one might be inclined to change the name of the theorem to *Galois' theorem* just as the name of *Pell's equation* has been changed by many to *Fermat's equation*. In the present case, there is, however, less substantial reason for making the change since Lagrange actually first used the theorem in a special but important case, while Pell had nothing to do with the equation which bears his name. The term *Lagrange's theorem* for the theorem in question was used already by J. Petersen, *De algebraiske Ligningers Theori*, 1877, and became widely known, especially through the German translation of this popular work under the title *Theorie der Algebraischen Gleichungen*, 1878. While it has been widely used it is not found in some of the standard works of reference, including the large German and French mathematical encyclopedias.

In view of the facts that the term *Lagrange's theorem* has appeared in some of the most frequently consulted group theory literature for such a long time, and that the workers in this field are naturally inclined to express their gratitude to Lagrange for his large share in bringing this theorem to the attention of the public, it seems to the present writer that the use of this term is appropriate. The student of the history of our subject is not likely to be misled thereby since he is constantly reminded of the fact that many mathematical terms have grown in richness of meaning with the development of our subject. Netto called the theorem in question "Lagrange's fundamental theorem of the theory of substitutions" in the opening sentence of an article published in the *Mathematische Annalen*, volume 13 (1878), page 249, and a similar term was used for it in the review of this article in the *Jahrbuch über die Fortschritte der Mathematik*. In Jordan's *Traité des substitutions*, 1870, page 25, there appears a heading entitled "Theorems of Lagrange and of Cauchy." It seems likely that this is the source of the term "Lagrange's theorem".

In closing we shall refer to a few conspicuous historical errors relating to group theory which appear in the *Fortschritte*. We do this not because it is unusually easy to find such errors here but because remarks relating to such a standard work are likely to be more effective than if they related to one less generally used. In an article published in the *Bulletin of the American Mathematical Society*, volume 27 (1921), page 459, we referred to a few mathematical errors relating to group theory and appearing in this work. The present remarks may be regarded as an addition thereto.[2]

[1] R. D. Carmichael, *Bulletin of the American Mathematical Society*, vol. 27, 1921, p. 474.

[2] A considerable number of other errors relating to group theory were listed by the present writer in an article published in this MONTHLY, *1913*, 14-20.

We begin with a review found in the latest available volume at the time this address was prepared—viz., volume 45, relating to the publications of 1914–1915. On page 254 thereof appears a review of an article by G. Heussel (misspelled in the *Fortschritte*) entitled "Ueber Gruppen aus zwei Elementen gleicher Ordnung, b und c, die der Bedingung $b^2 = c^{2m}$ genügen." It is at once evident that the letter m in this title is superfluous for if m is odd it must be prime to the order of c and if it is even this order must be the double of an odd number to which m must again be prime. Hence the article in question is devoted to the well-known groups which are generated by two operators having a common square. Moreover, it does not contain any really new theorem; but neither the author of the article nor the reviewer in the *Fortschritte* indicated that we are here dealing with well-known results expressed in a form which differs only slightly from that in which they appeared earlier. This is the more singular because an article on these groups by the present writer appeared about ten years earlier in the same journal in which Heussel published his article.

The special case when each of the two generating operators which have a common square is of order 2 was fully considered by J. de Perott in the *Bulletin de la Société Mathématique de France*, volume 21, 1893, page 62. This case is so simple that others must have recognized it at an earlier date but the *Encyclopédie des Sciences Mathématiques* fails to give any reference relating thereto. It may be noted, in passing, that the references in this encyclopedia relating to the well-known simple group of order 168, tome 1, volume 1, page 563, are misleading. It would appear from these references that Hermite and Brioschi first directed attention to this group while this honor actually is due to Kronecker and Mathieu. The volume in which Mathieu published his results bears an earlier date (1858) than the volume in which Kronecker's results were published (1859), but Kronecker's paper was presented several months before that by Mathieu, and hence it may be just to say that Kronecker first directed attention to this group, as is done in the *Fortschritte*, volume 21, page 142.

The honor of Kronecker in this connection is beclouded by his remarkable display of ignorance in regard to the then available literature relating to substitution groups since he stated, in substance, that the group of order 168 and degree 7 constitutes the first example of a substitution group of degree n and containing more than one regular cyclic subgroup of this order without being either alternating or symmetric. Among the then known substitution groups satisfying this condition is the triply transitive group of degree 6 and of order 120, studied by Hermite and Cauchy more than a decade earlier. The subgroup of order 60 contained in this group is also an example of such a group. The history of the simple group of order 168 is perhaps especially interesting since this is the simple group of next to the lowest composite order.

Hence it may be of interest to note that Brioschi stated incorrectly that two groups of degree 7 and of order 168 exist, *Annali di Matematica*, volume 2 (1859), page 60. It may also be desirable to add that none of these early workers on the simple group of order 168 seems to have recognized that he was actually dealing

with a simple group. The simplicity of this group seems to have been first established in Jordan's *Traité des substitutions*, as a special case. It is well known that Galois first directed attention to the simplicity of the icosahedral group, which is the simple group of lowest composite order. The history of simple groups of composite order has its origin in the works of Galois but finds its first considerable development in the works of Jordan.

The student who is inclined to accept statements found in standard works of reference without attempting to test them as regards accuracy might be referred to page 160 of volume 34 of the *Fortschritte*, where it is stated that W. Burnside proved that a group which admits an automorphism of order 3 has the property that every pair of conjugate operators contained in it are commutative. This is the opening sentence of the review in question and hence it is the more striking since it is so obviously incorrect. To the thinking reader such obvious misstatements on the part of a reviewer are often refreshing since such a reader will readily see that some condition must have been omitted and in supplying this he may secure a deeper insight into the theorem under consideration.

It is evident that the order of the group of isomorphisms of any group G is equal to the number of different possible ordered sets of k independent generators of G such that the αth operator of each of these sets corresponds to the αth operator of a given one of these sets in some automorphism of G where α is any one of the numbers $1, 2, \cdots, k$. In the special case when G is an abelian prime-power group and its k independent generators are a reduced set, all the operators of the same order in such a set of generators may correspond, and the situation becomes unusually clear. It seems questionable whether such obvious facts should be noted in a review as if they were results found in the article under review, as is done in the *Fortschritte*, volume 40, page 192. What is, however, more to the point is that the term *invariant* is used in this particular article, and the review thereof, with an unusual meaning, and the reading of the review is made more difficult thereby as well as by other obscurities which are not removed in the review.

We venture to refer here to a somewhat amusing incident related to the subject under consideration. The able mathematician I. Schur of Berlin stated correctly on page 189 of volume 40 of the *Fortschritte* that the present writer attributed to him an assertion which he did not make and which, moreover, is false. On page 177 of the following volume of the same journal Schur attributed, in turn, to the present writer an assertion which he also did not make and which, moreover, is also false. It is stated here that the present writer asserted that when the Sylow subgroups of order p^m are transformed according to a triply transitive group then the prime number p must exceed 2. That this assertion is false results directly from the fact that in the alternating group of degree 5 the Sylow subgroups of order 4 are evidently transformed according to a triply transitive group. When a reviewer uses somewhat harsh language about a mistake it is perhaps natural for the one who made this mistake to feel a slight relief when he observes that the reviewer's harsh language applies equally to one of his own mistakes even if this does not remove the stain.

The preceding references to reviews in the *Fortschritte* do not imply that the present writer does not believe in critical mathematical reviews. On the contrary, he heartily believes in public criticism since such criticism is much more apt to be fair than private criticism, and most people form their judgment of the works of others from one or the other of these two sources. It is unfortunate but natural that the reviewers for the *Fortschritte* have exhibited wide differences of judgment as regards such reviews. Many of them were non-critical and practically contented themselves with an exhibition of the main results found in the work under review without even distinguishing between what was known earlier and what was then new. A few others were unduly critical as a result of their own ignorance of the subject under review and their youthful ambition to appear wise. On the whole these reviews are, however, very valuable as many of us know from experience and the *Fortschritte* has always been one of the most important mathematical periodicals for those seeking a broad knowledge of our subject.

It was not without misgivings that I ventured to appear before you on such an occasion as a critic of certain minor parts of the literature to which so much of my energy has been devoted. The normal attitude of mind should be to get all the good we can out of the writings of others and to make allowances for shortcomings, for we all admire generosity and many of us feel the need thereof. On the other hand, as teachers it behooves us also to strive hard to guide those who depend on us so that they may avoid the pitfalls which beset their ways. It is, therefore, necessary for us to become also familiar with this less attractive phase of our subject. Criticisms do not necessarily imply that the author thereof feels that he could have done better than the one that is being criticized but only that he feels that he can add another element which will tend to make the truth stand out more brightly. The preceding remarks are to be construed in this spirit.

THE APRIL MEETING OF THE IOWA SECTION.

The tenth regular meeting of the Iowa section of the Mathematical Association of America was held, in conjunction with the thirty-sixth annual meeting of the Iowa Academy of Science, at Drake University, Des Moines, Iowa, on April 29, 1922. The meeting consisted of one session with Professor C. W. EMMONS presiding. There were twenty-five in attendance, including the following twenty members of the Association:

O. W. Albert, E. W. Chittenden, Julia T. Colpitts, C. W. Emmons, Fay Farnum, C. Gouwens, E. C. Kiefer, G. E. King, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, C. W. Wester.

The following officers were elected for 1922–1923: Chairman, C. W. EMMONS, Simpson College; Vice-chairman, I. F. NEFF, Drake University; Secretary-

treasurer, J. F. REILLY, State University of Iowa. A committee consisting of Professors Wester, Colpitts and Neff was appointed to draw up resolutions expressing the feeling of the section on the death of Professor B. F. Simonson, and the illness of Professor W. J. Rusk.

The next meeting will be held at Des Moines in November.

The following papers were presented:

(1) "The method of averages" by Professor G. W. SNEDECOR;

(2) "Mathematical bulletins" by Professor T. M. BLAKSLEE (by title);

(3) "Topics in general analysis" by Professor E. W. CHITTENDEN;

(4) "The place of the equation in a scheme of education" by Professor C. W. WESTER;

(5) "Persymmetric determinants whose elements are finite sums" by Professor E. R. SMITH;

(6) "On the meaning of the classification of statistical series by means of the Lexis ratio" by Professor H. L. RIETZ;

(7) "Definitions of imaginary and complex numbers" by Professor E. S. ALLEN (by invitation);

(8) "An extension of the figurate numbers" by Professor J. F. REILLY.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. The generalization of the King-Hardy method of fitting the mortality curve was indicated. The method of averages is the special case in which the function to be fitted is linear. This method was compared to the method of least squares. Professor Snedecor showed that the method of averages can be depended upon to give good results if, and only if, the departure of the data from linearity is small.

2. Professor Blakslee suggested a solution of a pedagogical problem. He thinks that in our teaching the earlier foundations of mathematics are usually omitted. He suggested that state bulletins be issued to remedy this, holding that under the plea of "saving time" time is now being lost.

3. Professor Chittenden presented, in addition to a discussion of the significance of the fundamental notions of neighborhood, limit and distance in general analysis, the solution of a problem proposed by Fréchet. If it is possible to define distance for a space S with a preassigned definition of elements of accumulation so that the elements of accumulation remain unchanged and every sequence of elements of S which satisfied the Cauchy condition has a limit in S, the space S is said to be complete. It is found, in answer to an inquiry of Fréchet, that the rational numbers form an incomplete space S.

(4) This paper asserts that the only equations needed to solve the usual problems of the grades are of the types, $n \pm a = b$, $an = b$, $x^2 \pm a = b$, $1/x + 1/a = 1/b$. The linear types can be thoroughly mastered for whole numbers in the second and third grades, and the extension to all rational numbers in the later grades is extremely easy. The work for teacher and pupil will be less than at present and with greatly increased efficiency. The equation will become,

for all educated people, the normal medium for thought about number relations and for expression of that thought. These statements were made by Professor Wester on the basis of tests given in the classroom.

5. In fitting parabolic curves by the method of least squares or by the method of moments a normal system of equations is formed whose determinant was given. Professor Smith showed that for certain useful sets of values this determinant may be reduced to a simple expression whose numerical value is easily computed.

6. In this paper Professor Rietz gave an exposition of the meaning of the Lexis ratio as it arises in the comparison of Poisson and Lexis distributions with corresponding Bernoulli distributions. He then showed how to apply this criterion to experiments with games of chance and to actual statistical data, and emphasized that we may obtain from this criterion important indications as to whether the dispersion has its foundation in a constant probability or in probabilities with certain types of variation.

7. Professor Allen's paper has been published in this MONTHLY, *1922*, 301.

8. In this paper Professor Reilly represented by ${}^aH_m{}^{a-k}$ the sum of all the products of degree m that can be formed from the quantities a, $a-1$, $a-2$, $\cdots$, $a-k$, putting ${}^aH_0{}^{a-k} \equiv 1$. Then he showed that if m and k be each given the values 0, 1, 2, 3, $\cdots$, a being arbitrary, a system of numbers is obtained similar to but more general than the figurate numbers. The law of formation of this system of numbers is

$$ {}^aH_{m+1}{}^{a-k} \equiv {}^aH_{m+1}{}^{a-k+1} + (a-k){}^aH_m{}^{a-k}. $$

These numbers were found useful in reducing a certain type of determinant.

JOHN F. REILLY, *Secretary-Treasurer.*

THE MAY MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The eleventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Association was held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 13, 1922. The Chairman of the Section, Mr. O. S. Adams, presided at both morning and afternoon sessions.

There were sixty-two in attendance, including the following forty-two members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, H. G. Avers, E. A. Bailey, A. J. Barrett, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, Paul Capron, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, J. B. Eppes, Harry English, J. N. Galloway, W. M. Hamilton, W. E. Heal, P. E. Hemke, Nelle L. Ingels, R. P. Johnson, L. S. Johnston, W. D. Lambert, J. J. Luck, E. S. Mayer, Frank Morley, C. A. Mourhess, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, M. R. Richardson, H. M. Robert, Jr., R. E. Root, W. F. Shenton, C. A. Shook, T. McN. Simpson, Jr.

Those in attendance at the meeting were guests of the Department of Mathematics of the Naval Academy for lunch. At the close of the afternoon session the program was varied by visits to points of interest about the Naval Academy, or to some of the athletic contests in progress there.

The officers elected for the year 1922–1923 are: FRANK MORLEY, Chairman; G. R. CLEMENTS, Secretary-Treasurer; W. D. LAMBERT and T. McN. SIMPSON, Jr., additional members of the executive committee. The twelfth regular meeting of the Section will be held in December, 1922, probably in Washington, D. C.

The following papers were presented:

(1) "Self-inverse curves" by Professor FRANK MORLEY;

(2) "The significance of assumptions" by Professor T. McN. SIMPSON, Jr.;

(3) "Remarks on the theory of relativity" by Professor F. D. MURNAGHAN;

(4) "On Græffe's method for the numerical solution of algebraic equations" (illustrated) by Professor L. B. TUCKERMAN, Bureau of Standards (by invitation).

(5) "The parallax of the moon and the ellipticity of the earth" by Mr. W. D. LAMBERT.

Abstracts of the papers follow:

1. Professor Morley's paper was, in brief, as follows: An equation of an algebraic plane curve, when written in conjugate coördinates, gives immediately the images of points in the curve. The successive images give an expression for Green's function. The cases when the image closes are naturally of interest. Then a point on the curve, which is one of its own images, will have other images on the curve. Such a curve may properly be called self-inverse.[1] A principal theorem on such curves is that if there be an isogonal mapping such that n points of one plane correspond to a single point of another plane, then to a line of the second plane will correspond a self-inverse curve of the first plane.

2. Professor Simpson began by quoting President Eliot who has said that "the main reason for the painfully slow progress of the human race is to be found in the inability of the great mass of people to establish correctly the premises of an argument." All thought starts with assumptions; they may be explicit or implicit, recognized or unrecognized. Out of implicit and unrecognized assumptions sophistry is bred. Failure to recognize assumptions as such has been the source of much bitter controversy, builders having sought to preserve the scaffoldings of thought as part of the permanent structure. Science is essentially pragmatic; the question of truth is the question of the truth of assumptions and its test is the test of experiment and experience.

Mathematics is perhaps more explicit in its assumptions than any other subject introduced into the curriculum at an early stage. It is better suited than any other to develop the sense of the significance of assumptions as well as the logical sense. The social significance of mathematics is to lie perhaps in its contribution to the building of a social consciousness which shall recognize that the stabilization of our institutions rests ultimately upon our ability to know and to test our assumptions, and upon a willingness to revise them without partisanship, or bitterness, or distress.

[1] Self-inverse curves and surfaces have been termed by the French "anallagmatic"—EDITOR.

3. Professor Murnaghan's remarks were elementary and informal. They were directed to an exposition and criticism of several recent articles on relativity, showing in a simple manner just what mathematical problems the writers had set for themselves and to what extent satisfactory conclusions had been secured. Some work by Whitehead was examined with particular care.

4. Professor Tuckerman pointed out that the method for the solution of numerical algebraic equations published by Græffe in 1837 and developed and perfected by Encke in 1841, and later by Runge and other German mathematicians, is almost completely ignored in English and American text books. Illustrations using a computing machine and also a slide rule were given, both for real and complex roots. Attention was called to an erroneous statement, that "no reliable estimate of the error (of the roots) can be given." The computational errors of the method were discussed and a simple expression for the upper limit to the error in the modulus of any root was derived. For determination of the amplitude of the complex roots, Encke's method is preferable in the case of one or two pairs. When three or more pairs are present, a displacement of the zero is to be recommended. The method was extended by a simple process to equations with complex coefficients, and an illustration of the solution was given. The author believes that Græffe's method deserves more attention than it has received, and that in many cases it is to be preferred to the better known Horner's and Newton's methods, because of its generality, simplicity, and directness of attack.

5. Mr. Lambert's paper gave a brief account of the theory by which Sir Isaac Newton connected the force governing the motion of the moon with terrestrial gravity. When all necessary refinements are introduced (involving, for example, the ellipticity of the earth), the fundamental equation of the theory may be used to compute the parallax of the moon; the parallax so obtained may be called the "dynamic parallax." The dynamic parallax does not agree with the parallax determined by direct observation at Greenwich and Cape of Good Hope, unless a rather large ellipticity of the earth is assumed, namely, about 1/293 instead of 1/297, the value which is generally accepted as nearly correct. This discrepancy in the ellipticities is rather larger than can reasonably be attributed to observational error in the parallax or to uncertainties in the other assumed data of the fundamental equation. The explanation is probably to be looked for in the deflections of the plumb lines at Greenwich and the Cape, amounting sometimes to 30″ or more. Deflections there of about 10″ would reconcile the observed and dynamic values of the parallax for an ellipticity of 1/297. A computation of the effects of surface irregularities represented by the topography within radii of 2,500 miles of Greenwich and the Cape, with allowance for isostatic compensation, indicates deflections of the vertical in the right direction but by only about one fourth of the requisite amounts. The results, however, could be changed by changing the hypothetical distribution of isostatic compensation. A further partial explanation of the discrepancy might be found if we accept Helmert's determination of the figure of the earth as a triaxial ellipsoid. The detailed discussion will probably be published by the U. S. Coast and Geodetic Survey.

G. R. CLEMENTS, *Secretary-Treasurer.*

"STATISTICS" IN A MATHEMATICAL ENCYCLOPEDIC DICTIONARY.

By H. L. RIETZ, State University of Iowa.

In commenting on Professor G. A. Miller's article in this MONTHLY (*1918*, 383) giving meanings of *group* and *group theory*, Professor E. R. Hedrick, Chairman of the Association's Dictionary Committee, suggested (*1918*, 428) that further sample definitions be submitted. As it is not unlikely that there will be considerable difficulty in determining the extent to which terms from applied mathematics should be included in the proposed dictionary, the writer has been considering the question with special reference to the mathematics of statistics, probability, and insurance. As a result of such consideration, the following brief paper is submitted to explain the meanings of *statistics* and of the associated expressions,[1] *statistical data*, *statistical methods*, *theory of statistics*, *mathematical statistics*, *statistical probability*, and to suggest a list of terms and expressions from statistics, probability, and insurance that should probably be included in the dictionary. In the selection of the list of terms and expressions given below, the writer has been guided by his experience with seniors and first-year graduate students taking courses in statistics and actuarial theory, and has included only terms and expressions within the range of reading of such students.

Statistics (stá-tĭs'tiks), F. *statistique*, G. *Statistik*, I. *Statistica*, Sp. *estadistica*. The word statistics seems to be derived from the Latin *status*, used in the sense of a political state. Statistics is a comparatively new word. Its first occurrence in English thus far noted seems to be in J. F. von Bielfeld, *The Elements of Universal Erudition*, translated by W. Hooper, London, 1770. One of the chapters of this book is called *Statistics*, and the subject is defined as "The science that teaches us what is the political arrangement of all the modern states of the known world." The word occurs in 1787 with a somewhat changed meaning in the preface to E. A. W. Zimmerman, *A Political Survey of the Present State of Europe*. In this work it is stated that about forty years before the branch known as statistics was formed into a separate science in Germany. The German word *statistik* was used by Professor Achenwall of Göttingen in 1749, and the Latin, *statisticus*, was used at a somewhat earlier date. In *Meyers Konversationslexikon*, 6 ed., volume 18, under "Statistik", Schäzer (1735–1809), a pupil of Achenwall, defined: "Statistik ist stillstehende Geschichte; Geschichte ist fortlaufende Statistik."[2] In 1790 Sir John Sinclair stated in a letter to the Clergy of the Church of Scotland that "statistical inquiries" have been carried to a great extent in Germany, and adds that the expression "statistical inquiries" means "inquiries respecting the population, the political circumstances, the productions of a country, and other matters of state." *Statistics*, as thus used by German writers and by others in the eighteenth century, meant an exposition of the character of the state, and such expositions were usually verbal rather than numerical. With the growth of official numerical data, it was natural that numerical statements should begin to replace the verbal statements. Statistics thus gradually came to mean an exposition of the attributes of the state by numerical methods. Following this usage, the word next came to denote the figures used in such descriptions. Thus, the collections of numerical data were called statistics. This use of the word prevails at the present time but the data may refer to the state or to any other subject.

There is, however, an element in the meaning of the word statistics as used at the present time in the theory of statistics that is not necessarily involved in a collection of figures. Thus, a collection of 1000 numbers consisting

[1] These expressions should be listed alphabetically and references should be given to see their meanings under *statistics*.

[2] This citation is contributed by Professor A. J. Kempner.

of the number 5 written 1000 times would not constitute statistics. Numerical data known as statistics or *statistical data* have a certain element of variability. For example, statistics on the statures of men are variable from man to man. Statistics of social interest show great variability from individual to individual and from community to community. Statistics of meteorology show great variability from time to time and from place to place. The term statistics as used at present in the theory of statistics means numerical data that exhibit variability in individual items, where such variability is ascribed to a multiplicity of causes. See G. U. Yule, *Introduction to the Theory of Statistics*, London, 1922, pp. 1–5; H. Bruns, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, Leipzig and Berlin, 1906, pp. 1–17; P. M. H. Laurent, *Statistique Mathématique*, Paris, 1908, pp. 1–16; and E. Blaschke, *Vorlesungen über mathematische Statistik*, Leipzig and Berlin, 1906, pp. 1–8.

The expression *statistical methods* means methods which are suitable for the description and characterization of statistical data. In the development of the meaning of statistical methods there have been influences in operation from three sources—the calculus of probability, the preparation of life and monetary tables under the name "political arithmetic," and the collection of data to be used in the machinery of government and business.

An exposition of the principles on which statistical methods are based is called the *theory of statistics*. A set of mathematical propositions that relate to statistical methods is often called *mathematical statistics*.

The relation of mathematical statistics to the theory of probability may be indicated by saying that the general problem of mathematical statistics in its ideal form is to determine a system of drawings to be carried out with urns of fixed composition, in such a way that the results of the set of drawings lead, with a high degree of probability, to a table of values identical with the statistical data (cf. E. Borel, *Éléments de la Théorie des Probabilités*, Paris, 1910, p. 167). Mathematical statistics is thus one branch of the theory of probability (cf. articles on *statistics* in German and French encyclopedias of mathematics). In fact, *a posteriori* or inductive probability is sometimes called *statistical probability*. (See E. Czuber, *Wahrscheinlichkeitsrechnung*, volume 2, Leipzig and Berlin, 1921, p. 6.) The concept of statistical probability is involved whenever the properties of an aggregate are predicted or inferred by observation of a sample taken from the aggregate. Many such inferences are drawn by persons unfamiliar with mathematical statistics, and there is practically no doubt that many conclusions thus obtained are invalid.

Mathematical statistics aims to establish criteria that give numerical values to the degrees of confidence to be placed in such inferences. In the development of these criteria statistical probability means the limiting value, as s becomes infinite, of m/s, where m is the frequency of happening of the event in s trials. The existence of the limit is assumed. The applications of mathematical statistics cover a wide range of scientific and social interests. These applications include the whole theory of insurance, and have an important place in biology, anthropology, psychology, economics, and even in the more exact sciences of chemistry and physics. To give a notion of the variety of applications, we may cite the following: *Biometrika*, "a journal for the statistical study of biological problems"; E. L. Thorndike, *Educational Psychology*, New York, 1913–1914; H. L. Rietz and H. H. Mitchell, "On the metabolism experiment as a statistical problem" (*Journal of Biological Chemistry*, volume 8, 1910, pp. 297–326); E. Rutherford and H. Geiger, "The probability variations in the distribution of α particles" (*Philosophical Magazine*, series 6, volume 26, 1910, pp. 698–707); J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, New York, 1902; F. Y. Edgeworth, "On the application of probabilities to the movement of gas-molecules" (*Philosophical Magazine*, series 6, volume 40, 1920, pp. 249–272). Lists of references on the methods and theory of statistics are given in G. U. Yule, *Introduction to the Theory of Statistics*, London, 1922, and E. Blaschke, *Vorlesungen über mathematische Statistik*, Leipzig and Berlin, 1906. A bibliography of applications prior to 1904 is given in C. B. Davenport, *Statistical Methods*, New York, 1904.

The following terms and expressions drawn from the theories of probability, statistics, and insurance are tentatively suggested for inclusion in the proposed dictionary. It seems that any term or expression that occurs infrequently, and that would require an appreciable amount of space to give its meaning in the dictionary, may well be treated by giving one or more references to its definitions

and use in the literature. In accord with this view, the mark "ref." is placed after each term that may, in the judgment of the writer, be treated in this way. Doubtless a considerable number of other terms may be treated by giving a very brief statement and references. In the preparation of the following list, the writer is indebted to Professor E. L. Dodd for valuable suggestions and additions.

Accumulation of discount, ref.
Actuarial theory
Adjustment of data, ref.
Advowson, ref.
Aggregate mortality table
Allocurtic, ref.
Amortization
Amortization of premium, ref.
Annual rent
Annuity,
  apportionable
  certain
  complete
  continuous
  deferred
  due
  forborne
  immediate
  intercepted
  joint life
  last survivorship
  life
  perpetual
  reversionary
  survivorship
  temporary
A posteriori probability
A priori probability
Array
Arithmetical mean
Association, theory of, ref.
Assurance
Automatic policy loan
Average
Average deviation
Bayes rule, ref.
Benefit of insurance, ref.
Bernoulli series, ref.
Bernoulli theorem, ref.
Bias in sampling
Biometry
Capitalized cost
Charlier Coefficient of Disturbancy
Claims, death
Class
Class frequency
Class interval
Class mark
Coefficient of association
Coefficient of contingency, ref.
Coefficient of variability
Commutation columns
Commutation symbols
Complement of life, ref.
Complete expectation of life
Compound reversionary addition, ref.
Contingency coefficient, ref.
Continuous instalment
Convertible term
Copyhold, ref.
Correlation,
  multiple, ref.
  normal
  partial
  rank
  spurious
Correlation coefficient
Correlation ratio
Cost of insurance
Cumulative graphs
Curtate expectation of life
Death rate, central
Death strain
Decile
Dependent events
Differences,
  central
  finite
  order of
Discount
Dispersion
Dividend (in life insurance),
  annual
  deferred
Endowment, pure
Endowment insurance
Equally likely
Equated time
Equation of life,
  of payments
  of value
Error
Errors, theory of
Expectation of life,
  complete
  curtate
Expiry
Extended insurance

Extrapolation
Figurate numbers
Fluctuations in sampling
Force of discount
Force of interest
Force of mortality
Frequency curve
Frequency distribution,
  binomial
  Gaussian
  normal
  skew
Frequency polygon
Frequency surface
Gain and loss exhibit
Gauss curve of error
Geometrical mean
Geometrical probability
Gompertz's law, ref.
Goodness of fit
Graduation of data
Group insurance
Hardy's formula, ref.
Harmonical mean
Heteroclitic, ref.
Heterograde series, ref.
Heteroscedastic, ref.
Heteroscedasticity, ref.
Histogram, ref.
Historigram, ref.
Homoclitic, ref.
Homogeneity in statistics
Homograde series, ref.
Homoscedastic, ref.
Homoscedasticity, ref.
Homotyposis, ref.
Incontestible
Independent events
Index of abmodality
Index number
Industrial insurance
Initial expense
Insurable interest
Insurance,
  assessment
  capital redemption
  casualty
  fraternal
  life
  property
Interest,
  accumulative rate of
  compound
  continuously convertible
  effective rate of
  frequency of conversion of
  instantaneous rate of
  nominal rate of
  rate of
  remunerative rate of
  simple
International actuarial notation, ref.
Isocurtic, ref.
Isotropic, ref.
Joint life annuity
Joint life insurance
Joint life probability
Lag
Lapse
Last survivorship annuity, ref.
Last survivorship insurance, ref.
Least squares, ref.
Legal reserve insurance
Level premiums
Lexis ratio, ref.
Lexis scheme, ref.
Life interest
Lifetime, most probable
Limited payment policy
Loading
Logarithmic paper, ref.
Lorenz graph
Lubbock's formula, ref.
Makeham's laws
Mathematical risk, ref.
Mathematics of statistics
Maturity
Mean contingency, ref.
Mean deviation
Mean error
Mean square contingency, ref.
Mean value
Median
Method of moments, ref.
Mode
Modulus as a measure of dispersion
Moments
Mortality rate,
  instantaneous
Mortality table,
  aggregate
  select
  ultimate
Moving average
Mutuality
Natural premium
Net premium
Net valuation
Nominal interest
Non-forfeiture
Normal frequency curve
Normal law of error
Normal probability curve
Ogive
Old line insurance
Ordinary life policy

Paid up insurance
Panmixia
Participating policies
Percentile
Perpetuity
Poisson scheme, ref.
Poisson series, ref.
Poisson theorem, ref.
Policy,
  continuous instalment
  endowment
  limited payment
  ordinary life
  pure endowment
  term
Polychroic functions, ref.
Precision
Premium,
  annual
  gross or office
  natural
  net
  single
Probability,
  a posteriori
  a priori
  deductive
  empirical
  inductive
  statistical
  theory of
Probable error
Probable lifetime
Prospective method
Pure endowment
Quartile
Quartile deviation
Radix of a table
Random sampling
Range
Rank
Rating up
Regression, linearity of
Regression coefficient
Regression curve
Reinsurance
Renewable term insurance
Reserve,
  initial
  mean
  modified preliminary term
  net level premium
  preliminary term
  select and ultimate
  terminal
Retrospective method
Reversion
Reversionary annuity
Reversionary expectations of life, ref.
Rests
Root-mean-square-deviation
Sampling
Schedules
Sheppard's corrections
Simple and compound survivorship, ref.
Simple sampling
Sinking fund
Skew frequency curve
Skewness
Smoothing
Spearman's coefficient, ref.
Square root of mean square
Standard deviation
Statistical mechanics
Statistical methods
Statistical probability
Statistics,
  theory of
  vital
Stirling's theorem, ref.
Surplus of an insurance company
Surrender charge
Surrender value
Tchebycheff's theorem
Term policy
Term premium
Tetrachroic functions, ref.
Tontine
Total and permanent disability
Trend
Uniform seniority
Valuation of policies, see reserve
Valuation date of issue
Variant
Variate,
  graduated
  integral
Vie probable
Weighted arithmetical mean
Woolhouse's formula, ref.
Yule coefficient, ref.

GEORGE BRUCE HALSTED.

By FLORIAN CAJORI, University of California.

[For several years the MONTHLY has had no leading article concerning a deceased mathematician. An exception is now made in the case of one who in its earlier years assisted very materially in the financial support of the MONTHLY, and contributed extensively to its pages.

Dr. Halsted's articles in the MONTHLY were over fifty in number, and one of them, "Non-euclidean geometry, historical and expository," consisting chiefly of his translation of Saccheri's *Euclides Vindicatus*, extended as a serial through the first five volumes. Twenty of the articles were biographical sketches, usually accompanied by portraits, in the series which Professor Finkel established and carried on for a number of years. These sketches were as follows: Alasia (volume 9), Barbarin (15), Beltrami (9), Bolyai Farkas (3), Cayley (2), De Morgan (4), Frost (6), Hoüel (4), Klein (1), Lambert (2), Lie (6), Lobachevsky (2), F. Schmidt (8), D. M. Y. Sommerville (19), Sylvester (1 and 4), Tchebychev (2 and 5), Tucker (7), and Vasiliev (4). Most of Dr. Halsted's other articles dealt with topics of non-euclidean geometry or the foundations of geometry.

An appreciative sketch of Dr. Halsted, by Professor L. E. Dickson, appeared in the first volume of this MONTHLY, *1894*, 337–340; another by C. Alasia may be found in *Le Matematiche*, Città di Castello, volume 2; and others in B. F. Finkel's *Mathematical Solution Book*, fourth edition, 1902, and in *National Cyclopædia of American Biography*, volume 3, new edition. The first three are accompanied by portraits. Halsted was the first student in mathematics at the Johns Hopkins University and his insistence on a course in the theory of algebraic forms started the brilliant investigations that characterized Sylvester's work in Baltimore. This is related in Cajori's *Teaching and History of Mathematics in the United States*, 1890, pp. 264–266. We have already indicated the leading facts of Dr. Halsted's life (*1922*, 187); see also this issue of the MONTHLY, page 352. A fairly complete list of his writings may be derived from: Sommerville's *Bibliography of Non-Euclidean Geometry*, 1911; Poggendorff's *Biographisches-literarisches Handwörterbuch*, 1898 and 1904; and the Royal Society's *Catalogue of Scientific Papers*, volumes 10 and 15—EDITORS.]

On March 19, 1922, there passed away in New York City, at the age of sixty-nine years, a unique and picturesque figure among American mathematicians. One of the earliest pupils of J. J. Sylvester at the Johns Hopkins University, a student in Germany and an instructor at Princeton University, he was later for nineteen years professor of mathematics at the University of Texas. Leaving Texas in 1903 he was for nine years successively connected with several institutions, passing rapidly from one to another. His closing years were spent with a son at Greeley, Colorado.

Halsted's scientific activity may be said to have penetrated three fields: (1)

the translation, with commentaries, of noted foreign works, particularly of the great researches of Lobachevsky, Bolyai and Saccheri, and some of the popular writings of Poincaré; (2) studies in the logic of mathematics, particularly of geometry; (3) criticisms of the mathematical text-books of his day.

American mathematicians are indebted to Halsted for making the writings of the creators of non-Euclidean geometry accessible to them in the English language. His commentaries were always spicy and valuable, even though, as a historian, Halsted was not always able to maintain the attitude of an impartial judge. At his hands Gauss, for instance, received scant justice.

The most conspicuous of his efforts in logic was his *Rational Geometry*, the first edition of which, published in 1904, was based on David Hilbert's set of axioms of 1898. The book was widely and favorably reviewed, both in this country and abroad. Certain logical defects were pointed out, which caused Halsted to prepare a revised edition (1907) in which the words on the title-page of the first edition, "Based on Hilbert's Foundations," were dropped. This revised edition was well received and was translated into several foreign languages. Logical precision was the dominant motive in the preparation also of his earlier texts, chief of which are his *Elements of Geometry* (1885) and *Synthetic Projective Geometry* (1906).

Among critics Halsted ranked as the most outspoken American opponent of slip-shod methods of reasoning, such, for example, as occurred in the same-direction theory of parallel lines and in the unrestricted employment of hypothetical constructions. Perhaps rather over-confident of the correctness of his own views, his criticisms of mathematical writers, and sometimes also of university administrators, were at times so violent as to recoil upon himself. Nevertheless, his influence upon the teaching of mathematics in this country has been decidedly beneficial. Several of our most active mathematicians of the present time received inspiration while they were pupils of Halsted.

I myself met Halsted in person only two or three times at scientific meetings and then, each time, only for a few moments. But I was in occasional correspondence with him since 1888. When I first entered upon historical work, I received from him valuable suggestions and much-needed encouragement. About 1895 he read the geometrical part of the manuscript of my *History of Elementary Mathematics*. In a letter to me he made a most unsparing attack, called forth by one of my criticisms of Euclid's logic and a clerical error that I had made in copying a reference to Euclid. In my reply, I acknowledged the clerical error, and made a few changes and additions to my account of Euclid. I also endeavored to point out to him the slight unreasonableness of his attack. By return mail I received an answer, which indicated that the storm had passed and that Halsted had fully re-entered the atmosphere of serene cordiality. He did not indulge in flattery; his criticisms were honest and based on a firm grasp of fundamentals; his correspondence was stimulating.

Halsted had in his private library some rare books which he prized highly. He himself has published statements relating to his copy of Saccheri. His copy

of the first (1637) edition of Descartes' *Géométrie*, which, through the courtesy of Professor C. I. Palmer, the present writer examined two years ago, contained on the first leaf the signature of Letenneur and the following inscription:

"Offert à Monsieur J. J. Sylvester Souvenir affectioné de son dévoué Chasles. Feb. 1847."

Further on appears the entry:

"To Dr. Halsted, with the kind regards and all good wishes of J. J. Sylvester."
2d May 1893.

And at the end of the book is pasted a small sheet containing the following note written by Halsted himself:

"This book, *La Geometrie*, treasured in the family of Letenneur as gift of the author and by the great geometer and historian of geometry M. Chasles as the gem of his collection, was given by him to Sylvester as recompense for a great service and by Sylvester to his favorite pupil Halsted, by whom it was conveyed to Professor C. I. Palmer.

"The signature of Letenneur with his minute but characteristic rubric, that of Chasles after his glowing words and of Sylvester after his gracious lines

Make the book priceless.

G. B. H."

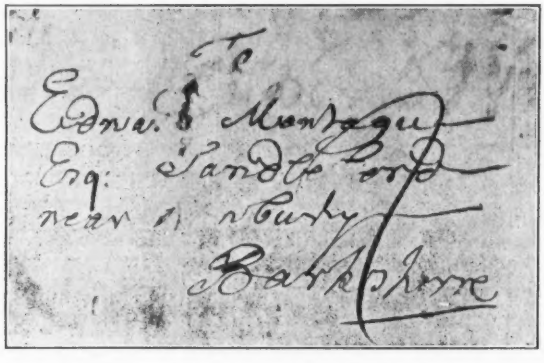
The spirit of Halsted has fled—gone to meet face to face the great masters he admired—Saccheri, Lobachevsky, Bolyai.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

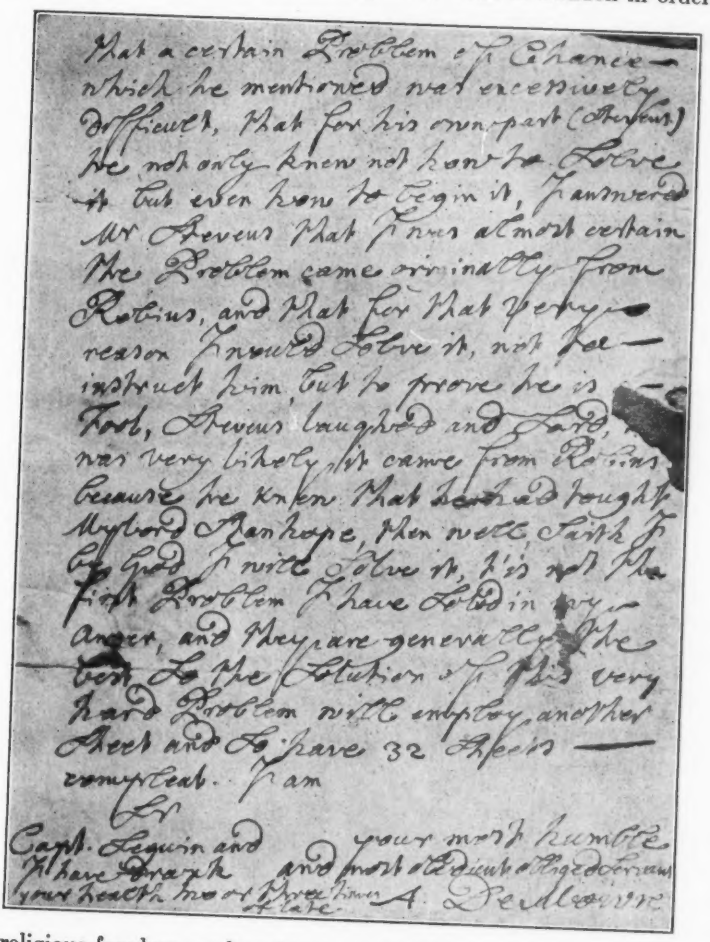
28. De Moivre Expresses Himself.

Of all the mathematicians who added to the reputation of England in the closing years of Newton's life, no one arouses a more sympathetic interest than Abraham De Moivre, author of the well-known *Doctrine of Chances*,[1] of the even



[1] London, 1718; second edition, London, 1738; third edition, posthumous, 1756.

more notable *Miscellanea Analytica*,[1] of a work on annuities, and one on series, and of various monographs on geometry and the Newtonian calculus. Born in France in 1667, he studied mathematics as a boy under Ozanam, one of the best teachers in Paris, and, at the age of eighteen, went to London in order to find



greater religious freedom, and perhaps greater adventure. Although compelled by narrow circumstances to forego the ideal life of a student, he became a member of the mathematical group of which Newton was the leader, and was recognized as a man of genuine ability. He made his living by teaching mathematics and by solving such problems as were brought to him by the curiously inclined, but such an existence is always a precarious one, and De Moivre's later years were

[1] London, 1730.

those of a rather lonely old man whose intimate friends had long since departed this world. He frequented Slaughter's famous coffee house in St. Martin's Lane, and that he imbibed something besides the drink from which the place received its name is not to be wondered, considering the period in which he lived and the circumstances which surrounded him.

Among my autographs is the last page of what was originally a four-page letter written by him to Edward Montague, a member of one of the most prominent of the noble families of England.[1] The letter is here reproduced in facsimile, being more legible than is usually the case with those that have come down to us from that period.

In this letter De Moivre speaks of a Mr. Stevens, probably Henry Stuart Stevens.[2] He also refers to a certain problem arising in the then popular game called "Hazard." This problem does not appear in the first edition of De Moivre's celebrated *Doctrine of Chances*, but it is No. XLVI of the edition of 1738. Todhunter,[3] in his *History of the Theory of Probabilities*, quotes De Moivre as saying:

"After I had solved the foregoing I spoke of my solution to Mr. Henry Stuart Stevens, but without communicating to him the manner of it. As he is a gentleman who, besides other uncommon qualifications, has a peculiar sagacity in reducing intricate questions to simple ones, he brought me, a few days after, his investigation of the conclusion set down in my third corollary; and as I had occasion to cite him before in another work, so I here renew with pleasure the expression of esteem which I have for his extraordinary talents."

Now by reference to the letter itself and to the above quotation and others from De Moivre, it is possible to reconstruct the probable facts of the case. This particular problem XLVI is "to find at Hazard the gain of 'the box' for any number of games divisible by 3," and is first given in the edition of 1738. In his *Miscellanea Analytica* of 1730, De Moivre refers to it, however, saying: "Septem aut octo abhinc annis D. Stevens Int. Templ. Socius, vir ingenuus, singulari sagacitate præditus id sibi propositum habens ut problema superius allatum solveret, hac ratione solutionem facile assecutus est, quam mihi his verbis exhibuit."

Now seven or eight years before 1730 would be about 1722–1723. But in the 1738 edition of the *Doctrine of Chances* De Moivre says that he solved the problem about twelve years before, which would be about 1726 (provided the revision was made the year the book was printed, which was probably not the case.) After De Moivre solved the problem, perhaps about 1724, he probably showed the solution to Stevens, who gave him a better one. We then have the date of the letter about 1723 or 1724.

[1] He was born in 1697 and was admitted as a Fellow of the Royal Society in 1746, having been elected the preceding year. He died in 1761. His signature may be found in facsimile in a book of the Royal Society, published in London in 1912.

[2] Born in 1707; died in 1782. He was elected to the Royal Society in 1740 and was admitted as a fellow the following year.

[3] Isaac Todhunter, born in 1820; died in 1884. He was successful as a writer of textbooks and his two historical contributions rank very high.

It should be stated, however, that it is by no means certain that the problem referred to in the letter is No. XLVI of the second edition of the *Doctrine of Chances*. It may be some other problem to which we have no certain way of attaching Stevens's name.

Coming now to the letter itself, it seems that the problem was given to De Moivre by Stevens about 1724; that De Moivre said that the problem very likely came originally from one Robins, probably Benjamin Robins,[1] to which opinion Stevens agreed; and that De Moivre had a rather poor opinion of Stevens at that time. The Lord Stanhope mentioned as having been tutored by Robins was doubtless Philip Dormer Stanhope, fourth Earl of Chesterfield, the well-known author of Chesterfield's letters, whose education was chiefly received through private instructors.

The reference in the letter to having worked out some of his best solutions "in my anger," might better, no doubt, have read "in my cups." The Captain Seguin, referred to in the post scriptum, is not mentioned in the standard works on biography. The "32 Sheets compleat" may refer to the revision of the *Doctrine of Chances*, already under way, or possibly to the *Miscellanea Analytica*. Books were not written in a hurry in those days.

The story, so often told, that De Moivre decided to sleep fifteen minutes longer each night, until finally he never awoke, is probably based upon some pleasantry of his in connection with his disease of somnolence which finally ended a life that had become quite unendurable, even with the help of such libations as were then poured out so freely.[2]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

The discussions which follow give cause for special satisfaction in that two of them are in some sense replies to previous papers, while the third closes with a question inviting further investigation. It would seem, from such indications, that this department is at times answering its purpose as a department of discussion.

The letter of Brigadier General Bixby, though it arises out of Professor Candy's article on mechanical solution of equations, is concerned more specifically with graphical solution, and emphasizes in a suitable way the great and increasing importance of this problem. In particular, the writer argues, from his experience, the advantages of Lill's graphical method, and recalls his own attempt in 1879 to make this method known to English reading students, by means of a pamphlet.

[1] Born at Bath in 1707; died at Fort St. David, Madras, in 1751; author of *Mathematical Tracts*, printed posthumously in 1761.

[2] I am indebted to Mr. Jekuthial Ginsburg for assistance in tracking down some of the facts in connection with this interesting letter.

On account of the great simplicity and utility of the method, and of its almost universal omission from texts in English, some readers of the MONTHLY will perhaps be glad to find a portion of General Bixby's pamphlet quoted at length.

The note by Professor Underwood completes the author's previous work (*1921*, 374–376), and bears some relation to his reply to Question **36** (*1922*, 255).

Professor Mathews proves directly a theorem regarding concyclic sets of points on an equilateral hyperbola which is the inverse of a theorem already known on a bicircular quartic.[1] If, in the notation of the paper, the points $\{B\}$ and $\{C\}$ are kept fixed, the sixteen circles AB_iC_jD through a point A define sixteen transformations of A into D. The question raised at the end of the paper is then how to determine the eight points $\{B\}$ and $\{C\}$ so that the transformations generate a finite group.

I. GRAPHICAL SOLUTION OF NUMERICAL EQUATIONS.[2]

By W. H. BIXBY, Washington, D. C.

I was much interested in the article of Professor A. L. Candy on mechanical solution of equations of the nth degree in a single variable.[3] Theoretically, the method is excellent; but practically there will be much difficulty in using it. It will be difficult to determine or secure the exactitude of the right-angles concerned, and the necessary length of the main bar may be awkward. The most perfect and practical mechanism for such equations is the multiple weighing machine, which has several graduated bars and sliding weights, the final weight indicating a root at each place where the machine balances. It would be costly, but no more so than the modern refined planimeters.

From my own personal experience in such matters, the method of Mr. Lill, Austrian engineer, developed by him about 1867 and exhibited by him at the Vienna World Exposition a little later, is the best graphical method yet developed, and far easier, quicker, and more exact, than any other graphical or mechanical method. I read of this about 1878, and published it in 1879 by a privately printed pamphlet.[4] At that date I had not seen Lill's 1867 printed article.[5] A few months ago I found that Luigi Cremona had also described Lill's method and made it public to English readers in 1888.[6] My pamphlet failed to attract much attention. A few engineers showed some interest in it; but appar-

[1] R. Lachlan, "On a theorem relating to bicircular quartics and twisted cubics," *Proceedings of the London Mathematical Society*, vol. 21, 1890, pp. 274–280; J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1915, p. 217.

[2] Extract from a letter to the editors.

[3] A. L. Candy, "A Mechanism for the Solution of an Equation of the nth Degree," this MONTHLY, *1920*, 195–199.

[4] W. H. Bixby, *Graphical Method for Finding the Real Roots of Numerical Equations of Any Degree if Containing but One Variable*, West Point, 1879.

[5] M. E. Lill, "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Nouvelles Annales de Mathématiques*, series 2, vol. 6, 1867, p. 359.

[6] L. Cremona, *Graphical Statics*, translated by T. H. Beare, Oxford, 1890, pp. 70–76. [More recent accounts of the method are found in C. Runge, *Graphical Methods* (Columbia University Lectures, 1909–1910), New York, 1912, pp. 11–12, and *Praxis der Gleichungen*, Berlin and Leipzig, 1921, pp. 101–110. See also this MONTHLY, *1911*, 159–162—EDITOR.]

ently there were only a very few who at that time had any occasion for solving such equations in their daily practice.

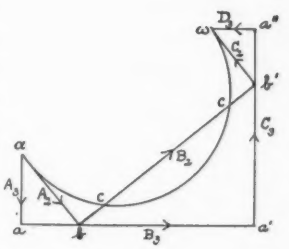
The following extract from the author's 1879 pamphlet will explain the method under discussion.[1] The proof may be easily supplied by the reader. The figure appended is for the case $n = 3$.

"Suppose any numerical equation

$$A_n x^n + B_n x^{n-1} + C_n x^{n-2} + \cdots + T_n x + U_n = 0,$$

in which A_n is any positive number, either whole or fractional.

"Commence, on a blank sheet of paper, at any assumed point α, and using any convenient scale, lay off in a downward direction a distance αa equal to A_n; through a, draw a perpendicular, and lay off upon it with the same scale as before, the value of B_n (laying off this distance to the right if B_n is positive, to the left, if B_n is negative); through the end of B_n, draw a perpendicular to B_n, upon which lay off the value of C_n, upward if positive, downward if negative; and so on. . . . [For each new line the positive direction turns through a right angle counter-clockwise.] Letter the end of the last line ω. We will then have a rectangular contour (that is, a broken line all of whose angles are right angles) of $n + 1$ sides commencing at α and ending at ω.



"Now starting again at α, draw at random any straight line cutting B_n in some point as b; through b draw a perpendicular to αb cutting C_n in some point as b', and so on; the result will be a new rectangular contour of n sides. If the nth side passes through the point ω, then $ab/\alpha a$, taken with its sign changed, is a root of the given equation. There will be as many such contours of n sides, and therefore as many points b, as there are real roots to the given equation.

"These rectangular contours can be readily determined by the aid of a ruler and right-angled triangle.

"Suppose only one root is found by the above method, giving us one new rectangular contour, $\alpha bb' \cdots \omega$, of n sides; call its first side A_{n-1}, its second B_{n-1}, and so on; this new rectangular contour, $\alpha bb' \cdots \omega = A_{n-1}, B_{n-1}$, etc., is the contour which represents the equation obtained by reducing the given equation to the $n - 1$ degree, by dividing out the root, $x = ab/\alpha a$, already found; treat this new contour of n sides like the preceding, obtaining a new rectangular contour of $n - 1$ sides whose first vertex is at some point c upon the line bb'; then $bc/\alpha b$, taken with its sign changed, will be another root of the given equation; and so on.

"The graphic solution of equations, by the above method of trial, will be an exceedingly simple process if the reader will proceed as follows:

"Procure a sheet of paper ruled in squares, each square being about a tenth of an inch on its side (such paper is easy to obtain, as it is nowadays in common use among surveyors and draughtsmen); procure also a sheet of tracing paper or cloth. Lay the tracing sheet upon the ruled paper, and, by the aid of the divisions upon the ruled paper, lay off on the tracing sheet the lines $ABCD$, etc.; thrust a needle, or pin, through the beginning of the A line and also through any convenient intersection upon the ruled paper; then, turning the tracing sheet about the needle as a pivot, the reader can readily follow by eye the ruled lines of the lower sheet, and can easily detect the auxiliary rectangular contours without having to pencil even a single unnecessary line.

"This graphical solution is especially applicable to cases where the desired roots lie between $\pm \frac{1}{2}$ and ± 5; if the desired roots lie beyond these limits, the given equation may advantageously be transformed into another whose roots will lie between the above limits."

In the case of a quadratic equation (represented by the inner contour) the

[1] For the extension of the method to complex roots and complex equations, see the article "d'après M. Lill" "Résolution graphique des équations algébriques qui ont des racines imaginaires," *Nouvelles Annales de Mathématiques*, series 2, vol. 7, 1868, p. 363—EDITOR.

two solutions, if real, may be found by means of a circle on $\alpha\omega$ as diameter, as indicated in the figure.[1]

II. Supplementary Note on the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Alabama Polytechnic Institute.

Theorem: *When an angle is rationally expressible in degrees and not a multiple of 30° or 45°, its trigonometric functions are irrational.*

The tangent of any rationally expressible angle except one of the form treated below may be proved irrational by a method so closely analogous to that used in my former paper that the details are omitted here. There remain angles of the form $(m/n)°$, where m/n is an irreducible fraction and where $m = k45$.

In the expansion

$$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{3!}\tan^3\theta + \cdots \pm \tan^n\theta \left(\text{or} \pm n\tan^{n-1}\theta, \text{ when } n \text{ is even}\right)}{1 - \frac{n(n-1)}{2!}\tan^2\theta + \cdots \pm n\tan^{n-1}\theta \text{ (or } \tan^n\theta, \text{ when } n \text{ is even)}}$$

the substitution $\tan n\theta = \pm 1$ gives an nth degree equation whose only possible rational roots (± 1) are excluded by the conditions of the problem. When $\tan n\theta = 0$ or $\pm\infty$, it is evident that the roots obtained by equating the numerator and denominator respectively to zero are either integers or the reciprocals of integers. Furthermore, if $\tan\theta$ is rational, $\tan 2\theta$, when it is finite, is rational. But if $\tan\theta = k$ or $1/k$, k being an integer, $\tan 2\theta = \pm 2k/(1-k^2)$, which is neither an integer nor the reciprocal of an integer, since the denominator lacks the factor k. Hence $\tan\theta$ is irrational.

The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers. Divide a right angle into n equal parts. Then there are only three distinct angles in all this infinite family which possess rational trigonometric functions.

Incidentally I have shown that the algebraic equations of the form (n odd)

$$x^{n-1} - \frac{n(n-1)}{2!}x^{n-3} + \cdots \pm n = 0,$$

and

$$nx^{n-1} - \frac{n(n-1)(n-2)}{3!}x^{n-3} + \cdots \pm 1 = 0,$$

have no rational roots.

[1] General Bixby would be glad to hear from any engineer, physicist or mathematician who is frequently required to solve higher equations, what method he uses, how much time is required, and what accuracy is deemed necessary.

III. Concyclic Points on an Equilateral Hyperbola and on its Inverses.

By R. M. Mathews, Wesleyan University.

The equation of an equilateral hyperbola may be written

$$xy - x - y = 0, \tag{1}$$

where the origin is a vertex and the axes of coördinates are parallel to the asymptotes. Writing $y = mx$, we can obtain the parametric equations

$$x = \frac{1+m}{m}, \qquad y = 1 + m. \tag{2}$$

There is one to one correspondence between all finite values of m (except zero) and all finite points on the curve.

These expressions (2) substituted in the equation of an arbitrary circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{3}$$

lead to the quartic equation

$$m^4 + 2(1+f)m^3 + (2 + 2g + 2f + c)m^2 + 2(1+g)m + 1 = 0. \tag{4}$$

The roots of this equation are the parameters of the points of intersection of the circle with the hyperbola. It is easy to show that the necessary and sufficient condition that four points of an equilateral hyperbola (2) be concyclic is that their parameters be such that

$$m_1 m_2 m_3 m_4 = 1.$$

Take three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ of parameters k_i, l_i, m_i, respectively $(i = 1, 2, 3, 4)$. The circle through A_i, B_i, C_i will cut the hyperbola in a fourth point D_i (parameter n_i) such that

$$k_i l_i m_i n_i = 1 \qquad (i = 1, 2, 3, 4).$$

From these equations, with the help of the facts that the sets of A's, B's and C's are severally concyclic, we obtain

$$n_1 n_2 n_3 n_4 = 1.$$

Thus we have proved the theorem:

Given three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ $(i = 1, 2, 3, 4)$ on an equilateral hyperbola; let the circle through $A_iB_iC_i$ cut the curve in a fourth point D_i; then the four points D_i are concyclic.

As specializations of this proposition we have two known theorems, the first due to Cazamian.[1]

Two circles meet an equilateral hyperbola in $\{A_i\}$ and $\{C_i\}$ $(i = 1, 2, 3, 4)$,

[1] "Sur l'hyperbole équilatère et sur ses inverses," *Nouvelles Annales de Mathématiques*, series 3, vol. 13, 1894, pp. 265–280.

respectively; draw the circle through A_i and tangent to the curve at C_i; the four circles so obtained meet the conic again in four concyclic points.

The four circles which osculate an equilateral hyperbola in four concyclic points cut the curve again in four concyclic points.

When an equilateral hyperbola is inverted about a point not on it, the transform is a bicircular quartic with a singular point at the center of inversion and the tangents there are at right angles (for they are parallel to the asymptotes of the conic). Thus the quartic is *orthotomic*. In particular, when the point of inversion is at the center of the hyperbola, the transform is a lemniscate. If the hyperbola be inverted about a point on it, the transform is a strophoid, oblique, in general, but a right strophoid, or Booth's logocyclica, when the center of inversion is a vertex of the conic. Every orthotomic bicircular quartic and every strophoid may be obtained by such an inversion.

As an inversion carries circles into circles, in general, the theorems just stated hold true when "equilateral hyperbola" is replaced by "orthotomic bicircular quartic" or by "strophoid," and the circles are conditioned as not passing through the node.

When one of the circles of the hyperbola passes through the center of inversion, special theorems result after the inversion. For example, in the main theorem for a strophoid let A_1 be at ∞, and then:

Let A_2, A_3, A_4 be three collinear points of a strophoid while $\{B_i\}$ and $\{C_i\}$ $(i = 1, 2, 3, 4)$ are two sets of concyclic points. Let the line B_1C_1 cut the curve in D_1, while the circle on $A_iB_iC_i$ $(i = 2, 3, 4)$ meets the cubic in D_i. Then the four points $\{D_i\}$ are concyclic.

In the case of one of the quartics the initial points may be arranged as: (1) one set of four collinear points with two sets of concyclic; (2) two sets of collinear points with one concyclic; (3) three sets of collinear points. None of the lines or circles of the quartic may pass through the singular point.

It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions.

On returning to the original theorem, we observe that there is no order assigned for the choice of a point triple to determine an (ABC) circle. When the three sets of concyclic points are assigned, it is evident that there are 64 circles (ABC) possible; each is determined by a triple of points, one from each set. Accordingly, there are 64 D-points which are concyclic in groups of four. To find how many of the possible ${}_{64}C_4$ are admissible here we observe: first, that each D-set is determined by a certain four of (ABC) circles; second, that in this set must be one circle for each of the A's; third, under this line of A's must be written some permutation of the four B's and some permutation of the four C's. Thus there are $\lfloor 4 \cdot \lfloor 4 = 576$ such sets and so 576 D-circles.

There are now $12 + 64 = 76$ points concyclic by fours in at least $576 + 64 + 3 = 643$ ways. It is easy to show that if a set of D-points be used with the B's and C's, the original A-set will be reached only once. It remains, then, to determine the conditions under which the configuration is closed as it stands, or will be closed after the addition of other points.

RECENT PUBLICATIONS.

REVIEWS.

Geschichte der Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. By J. TROPFKE. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger. Erster Band,[1] 1921, *Rechnen*, 8 + 177 pages; zweiter Band, 1921, *Allgemeine Arithmetik*, 4 + 221 pages; dritter Band, *Proportionen, Gleichungen*, 1922, 4 + 151 pages. Price \$.70 + \$2.00 + \$1.44.

Tropfke's *Geschichte* appeared first in 1902, as a two-volume work. It was well received as an interesting and worthy attempt at a systematic history of the fundamental ideas of arithmetic, algebra, geometry, and trigonometry.

The outstanding defect of the first edition was the reliance placed upon the authority of Cantor. This standpoint has been entirely changed in the present edition and the author has made every effort to found his conclusions upon source material. The preface indicates that the author has had the great advantage of the friendly criticism of Gustav Eneström, of Stockholm, and of H. Wieleitner, of Augsburg; no abler critics for work on the history of elementary mathematics could be found anywhere. Equally fortunate has Dr. Tropfke been in securing the aid of Julius Ruska, of Heidelberg, for points on Arabic mathematics and terminology.

The radical nature of the revision is only partly indicated by the fact that the first volume, *Rechnen* and *Algebra*, of the first edition required 332 pages, as opposed to a total of 549 pages now for these three sections, containing the corresponding material. More significant is the fact that the first edition of these sections included a grand total of slightly over 1200 footnotes, as opposed to over 2600 in the revised edition. Frequently, even yet, Tropfke cites Cantor when the original is easily available, as in respect to Egyptian duplication (note 251, volume 1) and in other places. The statement, page 52, volume 1, that the works of Jordanus had the widest circulation because of the cultivation of science largely by church schools is not all in accordance with historical fact. The works of Sacrobosco and Alexander de Villa Dei had the widest circulation, evidenced by the great number of manuscripts in European libraries, and these works also had the most enduring influence on subsequent works before the invention of printing; this conclusion has been definitely established in a thesis at Michigan, *A comparative study of the early treatises introducing into Europe the Hindu art of reckoning* (Concord, N. H., 1916) by Professor Suzan R. Benedict, of Smith College.

I note on page 13, volume 1, that the reference to the use of "Zeyfferzale" is incorrectly transcribed from the 1514 edition of Koebel's *Rechenbüchlein*. However, this is evidently only a slip as other citations checked are found accurately given.

The topics treated in the first part, of five parts planned, embrace numbers in general, including names and symbols, measures of time, of angles, and decimal

[1] Already reviewed in this MONTHLY, *1922*, 16–17.

measures, computation with integers, denominate numbers, properties of numbers, tables, fractions, and applied arithmetic. Under computation each fundamental operation is discussed with illustrations.

A notable omission is that form of division in which the remainders are written but not the partial products, commonly called the "Austrian method" and usually associated with the additive method of subtraction. Tropfke indicates that the complete "Austrian" process appeared first about the middle of the nineteenth century, but omission of partial products is found in Clavius (1603 ed. consulted), Manelli of 1659, Dechales of 1690, Malcolm's *A New System of Arithmetick* of 1730 and Sadler's *Complete System* of 1773; this method is designated as "a danda" or as Italian.

The second part of this work treats algebra and logarithms under similar detailed divisions. This method of treatment makes the work of great value to the teacher who wishes to find the historical development of any given topic of elementary arithmetic or algebra. It is highly to be recommended for this type of use.

The treatment of the algebraic symbols is given in a satisfactory manner; further light would have been obtained by consulting an article by Dr. Suzan Benedict on "The Development of Algebraic Symbolism," written at Teachers College, using the Plimpton collection and the private collection of D. E. Smith[1]. The statement (page 16) that the abbreviation used by Regiomontanus for *minus* is to be read "ig" is not at all accurate; the abbreviation is entirely regular for *minus*, the symbol like a 9 being standard for "us" and the "m" and "n" indicated by a bar above the "i."

So far as the computation with zero is concerned Tropfke leaves the impression that the early writers on algorism neglected this. However, both of the treatises published by Boncompagni (Trattati I and II), and those of Sacrobosco and Alexander de Villa Dei expressly mention this computation with zero.

As a late illustration of a mathematician who did not believe negative roots to be possible Tropfke cites Harriot's *Artis analyticae praxis*, 1631. A whole series of later illustrations is to be found in Maseres, *Tracts on Numerical Equations*.

Repeatedly in the treatment both of arithmetic and of algebra (*e.g.*, volume 1, pages 17–18; volume 2, pages 8–12; volume 3, pages 28, 36, 50, 112) Tropfke is compelled to treat the Hindu mathematics. Unfortunately Tropfke has taken the suspicions of Mr. G. R. Kaye concerning the non-Hindu origin of many mathematical ideas for fact. Historical things must be treated historically. Definite and concrete evidence has been found of numerous contributions to arithmetical and algebraical ideas by Hindu and Arabic scholars; any revision in favor of Greek origin must be based on historical evidence, not suppositions. Until the Greek documents are found no historian has any right to postulate the existence of Greek documents, and deny credit to Hindus, to Arabs, and to Babylonians solely on the basis of some hypothetical assumptions. Square root,

[1] *Teachers College, Columbia University, Circular, Department of Mathematics*, 1906–1907; *School Science and Mathematics*, vol. 9, 1909, pp. 375–384.

numerical symbols, negative numbers, the treatment of zero, fractions, algebraic equations, large numbers, and other related ideas are all topics which lead us historically to India; the whole is self-supporting and bears internal evidence of Hindu origin of more weight than any single document; but even the documentary evidence is available in mass.

Under *Allgemeine Arithmetik*, in the second volume, Tropfke treats in the first subdivision the history of our algebraic symbols and the introduction of general coefficients. The word algebra is discussed historically in a separate chapter. In a section devoted to the development of the concept of algebraic number Tropfke treats unity, zero, infinity, fractions, irrationals, negative numbers, and complex numbers. In the section devoted to the algebraic operations, the fundamental operations and also powers and roots are historically discussed. The second volume closes with a discussion of logarithms.

The historical development of topics connected with the theory of equations and of proportion is ably given in the third volume. The most notable omission seems to the writer of this review to be any mention in the discussion of quadratic equations of the algebra of Abu Kamil and of its influence upon the work of Leonardo of Pisa and Al-Karkhi [See my article in this MONTHLY (*1914*, 37–48), "The Algebra of Abu Kamil"; also in *Bibliotheca Mathematica*, volume 12, 1912, pp. 40–55]. To say that Al Karkhi followed only Greek models is nonsense; his work on quadratics is directly taken from Abu Kamil as I have shown. The date of Diophantos, 360 A.D., is too late by at least a century. The reference (page 42) to Greek normal forms of equations (*e.g.*, $x^2 = ax$, etc.) should be to Arabic forms.

The valuable list of problems and illustrations from classical authors has been increased, and an interesting appendix added on the Arabic treatment of the cubic equation.

Fortunately Tropfke has avoided, in general, the offensive national emphasis which so frequently mars the monumental work of Cantor. We see traces of Cantor's influence still in the references to Bürgi under decimal fractions in the first volume and under logarithms in the second volume.

This new edition is to be highly commended. Unfortunately no similar work is to be had in English, although even a translation of this work would be desirable. The scientific activity in Germany under the present undoubtedly distressing economic conditions arouses our admiration.

The prices in marks of books published in Germany vary almost from day to day; the publishers have affixed the prices indicated above. Notwithstanding the resentment which Americans may feel in the fact that these and other publishers in Germany make American prices higher than European it must in fairness be admitted that these prices are far less than recent English prices on volumes of this size. It is to be hoped that definite encouragement of this scholarly publication will be forthcoming from America, even though some of the orders go through German dealers to take advantage of the lower price in marks in Germany.

LOUIS C. KARPINSKI.

ARTICLES IN CURRENT PERIODICALS.

ALCALDE (Austin, Texas), volume 10, September, 1922: "George Bruce Halsted" by H. Y. Benedict, 1357–1359 ["Halsted was not a member of the first 1883–84 faculty of eight professors, but came the second year to succeed as Professor of Mathematics, William Le Roy Broun, who had gone back to Alabama to be president of the Alabama Polytechnic College. Halsted was for some time the first and only young 'professor,' all his colleagues being 'elder statesmen' who had already won their reputations in other institutions. The young, erratic, and somewhat bizarre Halsted presented quite a contrast to the older and more conventional Gould, Dabney, *et al.* During the 18 years that he remained in the University he was the most conspicuous and spectacular member of the faculty, at least so far as the students were concerned. There were noted men in the faculty and the students knew it, but they talked more of Halsted than of the others. This was because he was always saying or doing something peculiar or interesting. To this day the old-timers remember how he used to drive rapidly all about town a Shetland pony hitched to a wagon built of four toy wheels and a long plank, himself sitting astride of the plank which also carried three five-gallon coal-oil cans full of mash for his pigs. Let any student of the early years get to telling stories and you'll soon hear him tell, not without some exaggeration, anecdotes of Halsted—how he tried to feed his family exclusively on prickly-pear salad, how he built his house on stilts to avoid malaria; how, driving too rapidly, he thanked the policeman who was about to arrest him, for stopping his runaway horse.

"Halsted was a graduate of Princeton, where there is a Halsted Observatory donated by a member of his family, and at Johns Hopkins was a pupil of the great but eccentric Sylvester and a member of the small band there in 1876 that is usually regarded as the first real graduate school in the United States. For scientific research, for discovery of new natural laws, for philosophical speculation, for plucking some new fact out of the vast unknown, Halsted ever afterwards had an unabated zeal, an enthusiasm that communicated itself to others even while they smiled at his. 'Creation discovery,' 'the on go and the uprush of civilization,' 'nothing can take the place of contact with the living spirit of research,' were phrases ever on his tongue. The unknown fascinated him, conventions bored him, and he yearned to make discoveries.

"Although an unsystematic teacher who did not force his students to cover much ground, he was very interesting and at times elucidated complexities with startling clarity. His public lectures were extremely popular, and on interesting topics. In debate he was acute, and amusing. . . .

"Some of Halsted's phrases still cling after the passage of many years. 'The successful American,' he once wrote, 'is highly migratory; I have been all over the world.' At an ancient meeting of the Texas Academy of Science, at which Alexander Macfarlane, Professor of Physics, but known locally as 'Johnnie,' was arguing that the square root of minus one was a rotor or 90 degree turner, Halsted jumped to his feet and cried, 'It is no more 90 degrees than it is the left ear of a white elephant.'

"Much more could be written. Was he lucky in the pupils that he had, or did he inspire them to great things? That he had great pupils is certain: Len Dickson, M. B. Porter, R. L. Moore, G. W. Pierce, R. A. Thompson, Florence Lewis, and others. From any other institution comparable with the University of Texas, there has come in our generation no such group. . . .

"To 'George Bruce,' as we used to call him, it is scarcely fitting to say 'Rest in Peace.' In the other world, with William James and Elliot Cowes, with his beloved Bolyai and Lobatchevsky and Sylvester, he will find his happiness, not in rest, but in gazing curiously at the universe as it appears from the other side. Loving fine distinctions, splitting and resplitting the boundary line dividing convergent and divergent series, may not Halsted's spirit even now be seeking the Aum of the Buddhists, the Alpha and Omega of the Christians, the Alif of the Mohammedans?

> 'A hair divides the False from True,
> Yes; and a single Alif were the clue
> Could you but find it—to the Treasure House
> And peradventure the Master too.'"]

ANNALS OF MATHEMATICS, volume 23, September, 1921: "On matrices whose elements are integers" by O. Veblen and P. Franklin, 1–15; "An algorism for differential invariant theory" by O. E. Glenn, 16–28; "The general theory of cyclic-harmonic curves" by R. E. Moritz, 29–39; "More theorems on the complete quadrilateral" by J. W. Clawson, 40–44; "A theorem on cross ratios in the geometry of inversion" by J. L. Walsh, 45–51; "The condition for an isothermal

family on a surface" by J. K. Whittemore, 52–55; "The reversion of class number relations and the total representation of integers as sums of squares or triangular numbers" by E. T. Bell, 56–67; "Note on the term maximal subgroup" by G. A. Miller, 68–69; "Reducible cubic forms expressible rationally as determinants" by L. E. Dickson, 70–74; "Note on the Picard method of successive approximations" by D. Jackson, 75–77; "A fundamental system of covariants of the ternary cubic form" by L. E. Dickson, 78–82; "The modular theory of polyadic numbers" by A. A. Bennett, 83–90; "Some algebraic analogies in matric theory" by A. A. Bennett, 91–96; "Generalized conjugate matrices" by P. Franklin, 97–100.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, June, 1922: "The February meeting of the American Mathematical Society" by R. G. D. Richardson, 233–244; "A property of continuity" by D. C. Gillespie, 245–250; "Kinematics in a complex plane and some geometric applications" by A. Emch, 251–257; "Note on some results concerning Fermat's last theorem" by H. S. Vandiver, 258–260; "Convex distribution of the zeros of Sturm-Liouville functions" by E. Hille, 261–265; "Books on Fourier series" by C. N. Moore, 266–270 [Review of E. W. Hobson, *Theory of Functions of a Real Variable and the Theory of Fourier's Series* (vol. 1, Cambridge, 1921) and of H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals* (2d edition, London, 1921)]; Review by R. D. Carmichael of *Oeuvres de G. H. Halphen* (vol. 3, Paris, 1921), by A. R. Crathorne of J. Lipka, *Graphical and Mechanical Computation* (New York, 1918), by A. Dresden of K. Knopp, *Funktionentheorie* (2 Parts, Berlin and Leipzig, 1918–20), and by F. Cajori of L. Carnot, *Réflexions sur la Métaphysique du Calcul Infinitésimal* (vols. 1 and 2 Paris, 1921), 271–273; Notes, 274–277; New publications, 278–280—July, 1922: "The April meeting of the San Francisco Section" by B. A. Bernstein, 281–285; "The Easter meeting of the Society" by R. G. D. Richardson and A. Dresden, 285–302; "A report on the scientific work of the Chicago Section, 1897–1922" by A. Dresden, 303–307; "Eliakim Hastings Moore Fund" by A. Dresden, 307–309; "Note on the division of a plane by a point set" by E. W. Chittenden, 310–312; "Note on steady fluid motion" by S. D. Zeldin, 313–315; "Two books on analysis" by A. Dresden, 315–317 [Review of G. Vivanti, *Lezioni di Analisi Infinitesimale* (2d edition, Torino, 1920) and of E. Pascal, *Lezioni di Calcolo Infinitesimale* (Milano, Part I, 4th edition, 1919; Part II, 4th edition, 1918, Part III, 2d edition, 1918)]; Review by E. B. Stouffer of V. Kommerell and K. Kommerell, *Allgemeine Theorie der Raumkurven und Flächen* (vols. I and II, 3d edition, Berlin and Leipzig, 1921), by G. E. Wahlin of L. J. Mordell, *Three Lectures on Fermat's Last Theorem* (Cambridge, 1921), by H. Bateman of A. A. Robb, *The Absolute Relations of Time and Space* (Cambridge, 1921), by E. P. Adams of L. Rougier, *Philosophy and the New Physics* (Philadelphia, 1921), and by C. N. Reynolds, Jr., of K. M. Kohler, *Das Exzentrizitätsprinzip als Korrelat zur Relativitätstheorie* (Vienna, 1921), 317–319; Notes, 320–325; New publications, 325–328.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 46, June, 1922: "L'œuvre mathématique de Georges Humbert, quelques mots sur Camille Jordan" by H. Lebesgue, 220–223 [Extrait de la leçon inaugurale de mathématiques donnée au Collège de France le 6 janvier 1922. "Lorsqu'en 1912, Georges Humbert fut nommé professeur de mathématiques au Collège de France, ses travaux nombreux, variés et importants, qui lui avaient valu d'être élu membre de l'Académie des Sciences en 1901, avaient attiré sur lui, depuis longtemps, l'attention des mathématiciens du monde entier. Après 1912, malgré une longue maladie, son activité scientifique ne s'est pas ralentie; dans les derniers mois de sa vie, il rédigeait encore un Mémoire qui ne fut publié qu'après sa mort, dans le premier fascicule du *Journal de Mathématiques* de 1921. Georges Humbert a publié plus de 140 Notes et Mémoires, souvent étendus. . . . Dans toute œuvre véritable se révèle une continuité de pensée qui permet de grouper tout naturellement les divers Mémoires autour de quelques idées directrices, de quelques préoccupations dominantes. Dans cette esquisse rapide, je me bornerai au groupement qui s'impose tout d'abord: recherches antérieures à 1898, relatives à l'étude des fonctions algébriques; recherches postérieures à 1898, relatives à la théorie et à l'utilisation des fonctions abéliennes singulières. . . . Ici, dans cette chaire où j'ai le redoutable honneur de lui succéder, Humbert fut un professeur particulièrement apprécié, dont les leçons étaient impatiemment attendues d'un auditoire toujours fidèle. A mon avis, rien ne montre mieux les rares qualités d'Humbert que ses succès au Collège de France, car la tâche d'un professeur y est si difficile que, pour ma part, j'ai toujours été tenté de la déclarer impossible. On n'a pas le droit de n'y faire qu'un enseignement classique, fût-il excellent; il y faut un enseignement toujours en progrès, toujours renouvelé, toujours original; par le fond, si possible, tout au moins quant au groupement des matières et à la compréhension du sujet. . . . Comment Humbert a-t-il réussi à se rapprocher assez de cet idéal pour contenter toujours ses auditeurs? Le règle de conduite

qu'il a adoptée est à la fois habile et modeste; elle a été, de plus, fort utile aux progrès des études mathématiques en France. Au lieu de compter, comme il en aurait eu le droit, sur l'originalité de son esprit et la rapidité de sa compréhension, il a préféré s'appuyer sur la solidité et la précision de ses connaissances pour tout ce qui touche au domaine algébrique. Dans ce domaine, il a su trouver des sujets de Cours précis et variés, et cela d'autant plus facilement qu'il a continué à travailler exclusivement dans ce demaine et qu'il était constamment au courant de tout ce qui se publiait le concernant. En faisant ce choix, Humbert ne risquait guère de voir son enseignement faire double emploi avec quelque autre; les sujets étudiés par Humbert sont, en effet, presque tous en marge des questions soulevées par les principaux courants de la pensée mathématique pendant les trente dernières années. . . . Puisque ma tâche consiste, aujourd'hui, à parler de la chaire de Mathématiques du Collège de France et de ceux qui l'ont occupée le plus brillamment, j'ai la joie de pouvoir rendre hommage a l'illustre doyen des mathématiciens français, à M. Camille Jordan . . . . Voici toute une série de problèmes dont la liaison avec certains de ceux étudiés par Humbert n'est pas trop lointaine; quelle différence cependant entre les méthodes de ces deux éminents mathématiciens. Humbert utilise partout et toujours la fonction analytique, algébrique même, qu'il réussit ingénieusement à faire intervenir dans bien des questions, même dans des questions arithmétiques; à cet égard, il s'apparente à Poincaré. M. Jordan, suivant la voie ouverte par Galois, traite de questions, relatives par leur énoncé même au calcul algébrique, sans le secours de l'appareil analytique, à l'aide de raisonnements presque synthétiques. Dans ces raisonnements on procède toujours, si l'on veut dire, à une analyse; mais l'instrument qu'on y emploie est sans cesse variable, on le construit, on le modifie à chaque instant, et c'est en ce sens que les raisonnements sont synthétiques. . . . Les excursions faites par M. Jordan hors des domaines de recherches traditionnels avaient d'ailleurs fait réfléchir bien des mathématiciens; ceux-ci ont donné à la jeune école des encouragements précieux qui ont largement compensé les quelques reproches qu'elle a dû subir. Reproches bien injustes d'ailleurs, car, loin de mépriser le calcul, nous sommes persuadés que nos procédés synthétiques actuels céderont quelque jour la place à des procédés meilleurs, parce que plus analytiques; nous ne prétendons qu'à être les précurseurs et les annonciateurs peut-être d'un Viète ou d'un Newton qui introduira un symbolisme nouveau ou des notions nouvelles, peut-être d'un Descartes ou d'un Cauchy qui utilisera un symbolisme déjà connu et qui l'élargira pour des fins nouvelles, et nous soupirons après le nouvel algorithme qui, entre les mains d'hommes habiles comme l'était Humbert, donnera facilement et elégamment plus que ce que nous n'obtenons que péniblement et lourdement."]

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

The Mathematics Club of Hunter College, New York City.

[*1918*, 187; *1921*, 387.]

The following meetings were held in 1921–1922:

March 3, 1921: Reception to freshmen.

March 7: Business meeting.

March 21: "Graphical solution of problems" by Isabel Graves '23.

April 4: "Problems from Jones's *Mathematical Wrinkles*" by Monica Gilloran '21.

April 18: "Geometric forms in art and nature" (illustrated with lantern slides) by Professor Lao G. Simons.

May 19: Business meeting.

May 20: Election of officers, as follows: President, Sarah Karnis '22; vice-president, Edna Kramer '22; treasurer, Rose Charlon '22; publicity manager, Henrietta Olidort '23; secretary, Adele Matzke '22.

September 26: "Women mathematicians" by Adele Matzke '22; "Sonya Kovalevski" by Edna Kramer '22.

October 10–November 7: Mathematicians distinguished in other fields: "Omar Khayyam,

poet" by Sarah Karnis '22; "Joffre, soldier" by Esther Alfert '24; "Pythagoras, founder of the theory of music" by Mary Goldstein '21; "Leonardo da Vinci, artist" by Eleanor Wohl '22; "Descartes, philosopher" by Rose Charlon '22; "Gerbert, pope" by Isabel Graves '23; "Ben Ezra, rabbi" by Freda Berkowitz '25; "Charles L. Dodgson, author" by Nettie Oken '25; "Carnot, statesman" by Elsie Theurer '21.

November 28: "The mathematics of map-making" by Sarah Malkin '23.

December 12: "Shortcuts in arithmetic explained by algebra" by Minnie Levine '22.

January 9, 1922: "Mathematical games" by Henrietta Olidort '23.

February 27: House warming. Exhibition of hornbooks, reckoning pennies, swam pan, abacus, Jacob's staff, quadrants, geometric square, Babylonian tablets, pictures of mathematicians and old mathematics books.

March 2: Reception to freshmen.

March 13: "The calendar" by Grace Anselm '22.

March 27: "Scales of notation" by Isabel Graves '23.

April 10: "Scales of notation" (continued) by Esther Alfert '24.

May 7: Election of officers, as follows: President, Sarah Malkin '23; vice-president, Isabel Graves '23; secretary, Esther Alfert '24; treasurer, Bessie Schoenfield '24; faculty adviser, Miss Marcia Latham.

May 22: "Alignment charts" by Bessie Schoenfield '24.

(Report by Adele Matzke.)

The Mathematics Club of the State University of Iowa, Iowa City, Ia.

[*1920*, 29.]

The officers elected for the year 1921–1922 were: President, Ralph Kennon, Gr.; secretary, Eda Harper, Gr. The following meetings were held:

November 1, 1921: "The value and place of mathematics in modern civilization" by Professor H. L. Rietz.

December 1: "Paradoxes of Zeno" by Ralph Kennon, Gr.

December 15: "Scales of notation" by Howard Hughes '23.

January 12, 1922: "Euclid's parallel postulate" by Orley Brown, Gr.

February 2: "History of logarithms" by George Parks, Gr.

February 16: "The law of the mean" by Elmer Anderman '22.

March 9: "Hyperbolic functions" by Floyd Harper, Gr.

March 23: "Life assurance" by Ross Moyer '23.

April 6: "Solutions of the cubic" by Eda Harper, Gr.

April 20: "Curve tracing" by Harley Chandler, Gr.

May 11: "Comets" by Helen Starbuck '24.

(Report by Floyd Harper.)

The Mathematics Club of Vassar College, Poughkeepsie, N. Y.

[*1918*, 136, 456; *1919*, 264; *1920*, 78, 427.]

The officers elected for 1921–1922 were: President, Theresa McMakin '22; vice-president, Erneste Goodman '23; secretary-treasurer, Dorothy Brown '23 for first term, Harvia Wilson '23 for second term; members of executive committee, Professor Elizabeth B. Cowley, Allegra Hamilton '22.

October 1921: Picnic on "Sunset Hill" with mathematical charades and puzzles.

November 3: Address of welcome by the president. "History of the adding machine" by Helen Taylor '22; "Simple types of adding machines" by Erneste Goodman '23; "Modern adding machines" by Allegra Hamilton '22. Demonstration of certain adding machines.

December 1: "Mathematical education in Czecho-Slovakia" by Maria Podzimkova '22, formerly at the University of Prague.

February 9, 1922: "Chance and luck" by Sarah Eicher '22. The president read a letter from Amy Davison '21, past-president, a student at the Sorbonne, Paris, describing a course on relativity.

April 19: "Historical mathematical puzzle problems" by Allegra Hamilton '22. Professor

Cowley showed photostats of an Italian mathematical manuscript of the Middle Ages, containing three of these problems.

May 18: Indoor picnic. The following officers were elected for 1922-1923: President, Harvia Wilson '23; vice-president, Erneste Goodman '23; secretary-treasurer, Mary Hall '24; members of executive committee, Professor Cowley, Jeannette Kinne '23.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2986. Proposed by C. F. GUMMER, Queen's University.

A triangle is inscribed in a circle. The arcs into which it divides the circumference are bisected at points forming the vertices of a second triangle. A third triangle is derived in the same way from the second, and so on. Prove that each set of alternate triangles approaches a limiting position.

2987. Proposed by PHILIP FITCH, North Denver High School, Colorado.

A flexible chain of length l and uniform weight is fastened at one end to the ridge of a roof with pitch p and slant height L. If the eaves of the roof are at a height h from the ground and the coefficient of friction between the chain and the roof is μ, how long will it take, after releasing the chain, for the highest end to reach the ground?

2988. Proposed by PHILIP FRANKLIN, Harvard University.

Prove, geometrically, that if in an ellipse the tangent at P cuts the directrices in Z, Z' and the remaining tangents from Z and Z' to the ellipse meet at T, PT is normal to the ellipse at P. (An analytic proof is given in the *Journal of the Indian Mathematical Society*, vol. 13, 1921, p. 234.)

2989. Proposed by L. M. HOSKINS, Stanford University.

How should the following questions be answered, assuming that the place referred to is in latitude 34° 8′?

A building twelve feet high has been erected 49 inches south of our lot line. We desire to erect a wall on our line six inches in thickness. (*a*) How high can we build the wall and have it wholly within the shadow cast by the building? (*b*) How high can we build the wall and have it within the shadow cast by the building during the winter months?

2990. Proposed by R. M. MATHEWS, Wesleyan University.

If a circle be bitangent to a conic, its center is on one of the axes of the curve.

2991. Proposed by E. J. OGLESBY, New York University.

Sum the infinite series,

$$1+\frac{3x^2}{2!}+\frac{4x^4}{4!}+\frac{6x^6}{6!}+\cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

SOLUTIONS.

2894 [1921, 184]. Proposed by PHILIP FRANKLIN, Harvard University, AND E. L. POST, Columbia University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates*.

III. For every m-class there is at least one *conjugate m-class*.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry*, vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

SOLUTION BY THE PROPOSERS WITH IMPROVEMENTS SUGGESTED BY W. E. CLELAND, R. HARTSHORNE, and G. E. RAYNOR, Princeton University.

Examples to prove the independence of assumptions I to VII.

I. The elements $A\ B\ C\ D$ and the sets AB and CD.

II. The combinations of six letters taken three at a time.

III. The elements $A\ B$ and the set AB.

IV. The elements $A\ B\ C, X\ Y\ Z$ and the sets: $ABZ, ACY, BCX, AX, BY, CZ, XY, YZ, XZ$.

V. S is the null class, and there are no m-classes.

VI. S is the null class, and there are two null-m-classes.

VII. S contains an infinite number of elements, and there are an infinite number of m-classes formed as follows: Start with two m-classes $A_1\ A_2$ and $A_3\ A_4$. Form all the classes required by I. Form new classes $B_1\ B_2$, $B_3\ B_4$, which are to be conjugate to these last. Satisfy IV by inserting in the classes common elements, $C_1, C_2, \cdots$, never using the same element in more than one pair of classes. Form all the classes with these new elements required by I, and repeat the process.

THEOREM 1. *Every m-class contains at least two elements.*

For suppose one m-class contained a single element A. By III there would exist an m-class not containing A, and by VI it would contain an element B. By I there would be a class AB and by III, a conjugate class containing neither A nor B. Thus we would have two m-classes conjugate to the class with a single element A, contradicting IV.

THEOREM 2. *S contains at least four elements.*

By V there exists at least one m-class, which by theorem 1 contains at least two elements. By III there exists a conjugate m-class which also contains at least two elements.

THEOREM 3. *S contains at least six m-classes.*

The previous proof shows the existence of two conjugate m-classes, each containing at least two elements. Each of the elements of one of these m-classes, with one of the elements of the second of these classes, by I determines an m-class, and these m-classes, at least four in number, are all distinct, since if two coincided, by II it would coincide with one of our original classes, and these would not be conjugate.

THEOREM 4. *No m-class contains more than two elements.*

Let A and C be any two non-conjugate m-classes. By VII we may suppose that the number of elements in A, say p, is equal to or greater than the number of elements in C. Let the elements of A be $A_1, A_2, \cdots, A_p$, A_1 the element common to A and C.

Let $B_1, B_2, \cdots$, be the elements of B, the m-class conjugate to A, B_1 the element common to B and C. C then contains A_1 and B_1, and if there are any other elements in C we may call them $C_1, C_2, \cdots, C_k$, so that $k \leqq p - 2$.

Now if B_x is any one of the B's except B_1, the m-classes $B_xA_2, \cdots, B_xA_p$ will be distinct and no one of them can contain A_1 or B_1. Otherwise one of them would be the class A and contain B_x, or the class B and contain one of the A's, which is impossible, since A and B are conjugate

classes. One of them may be the conjugate of C, but each of the others, if $p > 2$, must contain one of the elements of C by IV, and that must be one of the C's and not A_1 or B_1. On the other hand, no two of them can contain the same C for they already have B_x in common. Therefore, there must be at least $p - 2$ of the C's, and, as k is not greater than $p - 2$, it follows that $k = p - 2$ and that C contains the same number of elements as A.

It follows also that one of these classes is the conjugate of C; that is, that the conjugate of C contains B_x, which is any one of the B's except B_1. This means that there can be only one B besides B_1, and that the class B can contain only two elements.

Now C was any class non-conjugate to A. Therefore, every class except B contains p elements while B contains only two. But if we start with some other class in place of A, we can prove that its conjugate contains two elements and that B contains p. This is possible only if $p = 2$ and every class contains only two elements.

THEOREM 5. *The elements of S and the m-classes are isomorphic with the elements A, B, C, D and the sets AB, AC, AD, BC, BD, CD.*

The four elements of theorem 2 may be labelled A, B, C, D and the six m-classes of theorem 3 the six of this proposition. By theorem 4 there are no more elements in any of these m-classes, and by II no more m-classes containing only these elements. Let X be an additional element. By I there is a class AX, and by theorem 4, it contains no more elements. We then have classes AX and AB both conjugate to CD, and since this violates IV it proves that there are no elements X. The six sets above evidently satisfy I to VII.

NOTE. In connection with the above solution Professor Veblen's comments on the origin and discussion of the problem will be of interest to the readers of the MONTHLY.

Professor VEBLEN says, "The problem originated in my course in Projective Geometry which I began, as usual, by a discussion of the abstract point of view in mathematics. In order to emphasize the point that our logical processes should be independent of any particular set of mental images or, indeed, of any knowledge of what the propositions are about, I proposed that certain members of the class should make up a set of postulates which would give the properties of some set of objects chosen but not divulged by them. The other students were then challenged to make logical deductions from the postulates and thereby deduce enough theorems to learn what the postulate makers had in mind.

"As a result of this suggestion, Mr. Post and Mr. Franklin brought forward a set of postulates which is essentially the one offered in the enclosed problem and a solution was found by several of the other students in precisely the manner indicated. One of the students indeed went further and pointed out that one of the postulates in the original set was redundant.

"The exercise was a success in showing how mathematical deductions can be made without knowledge as to what one is reasoning about. It also brought out vividly the problem of the significance of the logical processes as a method of discovery. While there is a sense in which it is true that you cannot get anything out of a set of postulates except what has been put in them, you can at least find out what was put in them. This is what the solver of this problem has to do in a simple case. In the more complicated case of ordinary geometry the student is apt to think that he understands what is in the axioms, but every time that he witnesses the derivation of a new theorem it turns out that there was something in them that he had not seen before.

"By a slight modification this problem can also be used to propose another problem which it seems to me may turn out to be an important one. Let us replace Assumption VII by the assumption that no m-class contains more than 6 elements. The mathematical science based on the assumptions could then be built up without ever counting beyond, let us say, 24. The question arises, how much of logic is needed to develop so limited a mathematical science? Would it be possible to single out a subset of the postulates of logic which would suffice for the purpose? If so, what processes of logic can be omitted? What sort of a logic results if the omitted processes are replaced by others? If it should turn out that the logic required for a satisfying theory of this finite system (or any other partciular system as, for example, a particular finite projective space) stops short of that required for larger systems, we would be in the presence of a criterion for the classification of logical processes which might help toward deciding the question as to what logical processes are legitimate in dealing with various types of infinite sets."

2899 [1921, 228]. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+ 60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

I. Solution by C. F. Gummer, Queen's University.

These are particular cases of a more general theorem. Consider n coplanar points $A_1, A_2, \cdots, A_n$; and let a point in the plane, starting at P, revolve successively about $A_1, A_2, \cdots, A_n, A_1, \cdots$ through angles $\phi_1, \phi_2, \cdots, \phi_n, \phi_1, \cdots$, until all the points $A_1, \cdots, A_n$ have been used m times each. The resultant displacement from P after the first set of rotations about $A_1, A_2, \cdots, A_n$ is the sum of a number of vectors, namely PA_1, A_1A_n and those obtained by turning A_nA_{n-1} through ϕ_n, $A_{n-1}A_{n-2}$ through $\phi_{n-1} + \phi_n$, $\cdots$ A_2A_1 through $\phi_2 + \phi_3 + \cdots \phi_n$ and finally A_1P through $\phi_1 + \phi_2 + \cdots \phi_n$. If Φ denotes the sum of the n angles, the vector sum after m sets of rotations contains two vectors of length PA_1 inclined at an angle $\pi + m\Phi$, and it contains m vectors of length A_iA_{i-1} (including the case of A_1A_n) inclined successively at the angle Φ so that they may be regarded as a series of equal chords placed end to end in a circle of suitable size. This is true for each value of i from 2 to n and for A_1A_n.

Three cases occur:

(1) $\Phi/(2\pi)$ *is rational but not integral.* A value of m other than 1 may then be found so that $m\Phi$ is a multiple of 2π. It follows that after m sets of rotations the two vectors derived from PA_1 are equal and opposite, and the vectors derived from A_iA_{i-1} are the sides of a closed regular polygon (possibly interlacing); therefore the resultant displacement vanishes, and the point has moved in a closed curve. The problems proposed belong to this case.

(2) $\Phi/(2\pi)$ *is irrational.* No group of vectors can be made to have a zero sum; and it will be found that the path cannot be closed (except for special positions of P). The path however lies in a finite region, since each group of vectors has a sum not greater than the diameter of its corresponding circle; and it may be shown that the path returns to positions indefinitely near to the initial point.

(3) $\Phi/(2\pi)$ *is an integer.* On taking $m = 1$, A_1P and PA_1 again destroy one another; but the remaining vectors, being now in groups of one, have not generally a zero sum. The resultant transformation is a translation of the entire plane, which by repetition (unless it happens to vanish) carries every point to infinity.

II. Solution by A. A. Bennett, University of Texas.

The problem will be treated in the more general case as follows: Given m and n two positive integers each greater than unity. Let $A_1, A_2, \cdots A_m, B_1$ be any $m + 1$ coplanar points. Let P describe a curve starting from B_1, made up of circular arcs, each of which is a one-(mn)th part of a circumference but with various radii as follows: With center A_1 describe one-(mn)th part of a circumference, positively from B_1, terminating at B_2. With center, A_2, describe one-(mn)th part of a circumference positively from B_2, terminating at B_3. Continue cyclically taking as the $(m + 1)$st center A_1, $(m + 2)$nd center A_2, etc. Show that B_{mn+1} coincides with B_1.

Let us use vector methods and denote the positive turn of one-(mn)th part of a circumference by the operator T. Then $T^{mn} = 1$. We shall then have the following relations: [1]

$$\begin{aligned} B_2 - A_1 &= T(B_1 - A_1), \\ B_3 - A_2 &= T(B_2 - A_2), \\ &\cdots\cdots\cdots \\ B_{k+1} - A_k &= T(B_k - A_k), \\ &\cdots\cdots\cdots \end{aligned}$$

Collecting terms,

$$B_{m+1} = T^mB_1 + (1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m)$$

and finally,

$$B_{mn+1} = T^{mn}B_1 + (1 + T^m + T^{2m} + \cdots + T^{(n-1)m})(1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m).$$

Since $T^m \neq 1$, and $(T^m - 1)(1 + T^m + T^{2m} + \cdots + T^{(n-1)m}) = T^{mn} - 1 = 0$, it follows that

[1] In this notation we may regard the difference of two points as a vector and the sum of a point and a vector as a point after the manner of Grassmann (see E. W. Hyde, *The Directional Calculus*, Boston, 1890, p. 2), but when we apply the distributive law to T, writing $T(B - A)$, for example, as $TB - TA$, we must understand A and B to represent vectors drawn from some arbitrary point O; that is, we may say that TB and TA stand for $T(B - O)$ and $T(A - O)$.

the operator $(1 + T^m + T^{2m} + \cdots + T^{(n-1)m})$ is a null-operator, so that

$$B_{mn+1} = T^{mn}B_1 = B_1$$

as desired.

It is of interest to carry out the work for an infinite number of points.

Let $x(t)$, $y(t)$ be the coördinates of a closed curve as the parameter t ranges from zero to 2π and be periodic with period 2π. Let $X(t)$ and $Y(t)$ be the coördinates of a derived curve obtained by a limiting process from the above discussion. Let us describe an infinitesimal circular arc with $x + \Delta t \cdot x'$, and $y + \Delta t \cdot y'$ as coördinates of the center, starting from the point with coördinates $X(t)$ and $Y(t)$, and described positively with an angle equal to the circumference divided by $n/(\Delta t/2\pi)$. The terminal point of the arc will be denoted by $(X + \Delta t \cdot X', Y + \Delta t \cdot Y')$. Since the arc is circular, we shall have

$$[(Y + \Delta t \cdot Y') - (y + \Delta t \cdot y')]^2 + [(X + \Delta t \cdot X') - (x + \Delta t \cdot x')]^2 = [Y - (y + \Delta t \cdot y')]^2 + [X - (x + \Delta t \cdot x')]^2. \tag{1}$$

The slope of the initial position of the radius is $[Y - (y + \Delta t \cdot y')]/[X - (x + \Delta t \cdot x')]$, and of the terminal position is $[(Y + \Delta t \cdot Y') - (y + \Delta t \cdot y')]/[(X + \Delta t \cdot X') - (x + \Delta t \cdot x')]$. The tangent of the angle of rotation is to be equal to $\tan(\Delta t/n)$; thus,

$$\frac{\dfrac{(Y + \Delta t \cdot Y') - (y + \Delta t \cdot y')}{(X + \Delta t \cdot X') - (x + \Delta t \cdot x')} - \dfrac{Y - (y + \Delta t \cdot y')}{X - (x + \Delta t \cdot x')}}{1 + \dfrac{(Y + \Delta t \cdot Y') - (y + \Delta t \cdot y')}{(X + \Delta t \cdot X') - (x + \Delta t \cdot x')} \cdot \dfrac{Y - (y + \Delta t \cdot y')}{X - (x + \Delta t \cdot x')}} = \tan\left(\frac{\Delta t}{n}\right). \tag{2}$$

Simplifying and dropping higher powers of Δt, we have from (2)

$$\frac{\dfrac{Y - y}{X - x}\left[1 + \Delta t\left(\dfrac{Y' - y'}{Y - y} - \dfrac{X' - x'}{X - x}\right)\right] - \dfrac{Y - y}{X - x}\left[1 - \Delta t\left(\dfrac{y'}{Y - y} - \dfrac{x'}{X - x}\right)\right]}{1 + \left(\dfrac{Y - y}{X - x}\right)^2} = \frac{\Delta t}{n}$$

or

$$Y'(X - x) - X'(Y - y) = [(Y - y)^2 + (X - x)^2]/n. \tag{3}$$

From (1), we have similarly,

$$Y'(Y - y) + X'(X - x) = 0; \tag{4}$$

whence,

$$-nX' = Y - y, \qquad nY' = X - x. \tag{5}$$

Eliminating between these we have the following pair of equations, to determine $X(t)$ and $Y(t)$, in terms of $x(t)$ and $y(t)$:

$$n^2X'' + X = x + ny', \qquad n^2Y'' + Y = y - nx'. \tag{6}$$

These are to be taken subject to the initial condition that when $t = 0$, $X(0)$ and $Y(0)$ have assigned values, and, from (5)

$$-nX'(0) = Y(0) - y(0), \qquad ny'(0) = X(0) - x(0). \tag{7}$$

The solutions are therefore determined.

Since $x(t)$ and $y(t)$ are periodic of period 2π, and since the solution of the homogeneous equations, $n^2X'' + X = 0$, $n^2Y'' + Y = 0$, are periodic of period $2n\pi$, it follows for this example that X and Y have each the period $2n\pi$. Thus for this "infinite" case also the set of derived points closes after the original set is described cyclically n times. It is to be noted that for each new choice of a variable t, the given curve whose parametric equations are $x = x(t)$, $y = y(t)$, is regarded as the limit of a new finite set of points.

NOTE BY THE EDITORS.—Professor Gummer's solution differs from the first part of Professor Bennett's only in that the rotations are different, making his solution more general.

These solutions may also be expressed in terms of complex quantities without any introduction of the notion of vectors or any operator T.

Also solved by T. M. BLAKSLEE and F. L. WILMER. Professor Blakslee sent in four different solutions, one of which was the generalized solution.

2893 [1921, 184]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two given skew lines in a variable plane turning about a fixed axis, not coplanar with either of the given lines.

SOLUTION BY J. K. WHITTEMORE, Yale University.

Let the axis of the variable plane be the z axis, and suppose the x and y axes chosen parallel one to each of the given skew lines. This coördinate system is generally oblique. The equations of the skew lines are (1) $y = b_1, z = c_1$; (2) $x = a_2, z = c_2$; the variable plane, $y = \lambda x$, intersects the two lines in points of coördinates, (1) $x = b_1/\lambda, y = b_1, z = c_1$; (2) $x = a_2, y = \lambda a_2, z = c_2$. The coördinates of the mid-point of the segment, a point of the required locus, are

$$x = \frac{a_2}{2} + \frac{b_1}{2\lambda}, \qquad y = \frac{b_1}{2} + \frac{\lambda a_2}{2}, \qquad z = \frac{c_1 + c_2}{2}.$$

Eliminating the parameter λ, the equations of the required locus are

$$\left(x - \frac{a_2}{2}\right)\left(y - \frac{b_1}{2}\right) = \frac{a_2 b_1}{4}, \qquad z = \frac{c_1 + c_2}{2}.$$

The locus is a hyperbola lying in a plane parallel to both given skew lines and half way between them; its asymptotes are parallel to the given skew lines; its center is the point of intersection of the diagonals of the parallelopiped, three of whose edges lie on the axis of the variable plane and the given skew lines; it intersects the axis of the variable plane.

Also solved by WILLIAM HOOVER.

2900 [1921, 277]. Proposed by I. A. BARNETT, University of Saskatchewan.

AB is the diameter of a circle and Q_0 any point on the circumference; $Q_1, Q_2, Q_3, \cdots$, are the points of bisection of the arcs $AQ_0, AQ_1, AQ_2, \cdots$; to prove that the product of the chords of the circle $BQ_1, BQ_2, BQ_3, \cdots, BQ_n$ is equal to $OA^n \cdot (AQ_0/AQ_n)$, O being the center of the circle.

SOLUTION BY A. M. HARDING, University of Arkansas.

Arc AQ_1 = arc Q_1Q_0. Hence, $\angle OBQ_1 = \angle AQ_0Q_1 = \angle Q_1AQ_0$. Hence, the isosceles triangles, OBQ_1 and Q_1Q_0A are similar. Then,

$$\frac{BQ_1}{AQ_0} = \frac{OB}{AQ_1}; \qquad \text{that is,} \qquad BQ_1 = OA \cdot \frac{AQ_0}{AQ_1}.$$

In a similar manner, it may be shown that

$$BQ_2 = OA \cdot \frac{AQ_1}{AQ_2},$$

$$BQ_3 = OA \cdot \frac{AQ_2}{AQ_3},$$

$$\vdots$$

$$BQ_n = OA \cdot \frac{AQ_{n-1}}{AQ_n}.$$

Multiplying these equations gives

$$BQ_1 \cdot BQ_2 \cdot BQ_3 \cdot \cdots BQ_n = OA^n \cdot \frac{AQ_0}{AQ_n}.$$

Also solved by T. M. BLAKSLEE, ARTHUR PELLETIER, and A. V. RICHARDSON.

2945 [1922, 29]. Proposed by T. M. BLAKSLEE, Ames, Iowa.

A point P in the plane of the triangle ABC rotates in a given direction around the vertices taken in either cyclical order, in each case through an angle equal to the corresponding angle of the triangle. That is, for example, AP rotates around A through an angle equal to the angle A of the triangle; then BP around B through an angle equal to the angle B, and so on. Prove

that P coincides with its original position at the end of six of these rotations. (See problem 2899, *1921*, 228.)

This is completely solved in Professor Gummer's solution of 2899 given above. Also solved by J. B. REYNOLDS and the PROPOSER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

R. M. DEMING, instructor of mathematics at Case School of Applied Science, has been appointed professor and head of the mathematics department at Upper Iowa University, Fayette, Iowa.

Mrs. W. C. GRAUSTEIN née CURTIS (*1921*, 334), formerly assistant professor of mathematics at Wellesley College (*1920*, 382), is again to become a member of the department there for 1923–1924.

GIUSEPPE LAIS, vice-director of the Vatican observatory since 1889, died December 26, 1921. He was born at Rome, April 15, 1845. Most of his meteorological, astronomical, and astrographical publications are to be found in *Nuovi Lincei, Atti*, Rome, from 1875 on. A biographical sketch and portrait are given in *Archivio di Storia della Scienza*, May, 1922.

ORAZIO TEDONE, born at Ruvo di Puglia, Italy, May 10, 1870, was killed in a railway accident April 14, 1922. He received his doctorate from the University of Pisa in 1892 and was a correspondent of the Academy of the Lincei. In 1899 he was appointed professor of higher analysis in the University of Genoa; in 1903 his title was changed to that of professor of rational mechanics. To volume IV–4 of the *Encyklopädie der mathematischen Wissenschaften* he contributed (1906): "Allgemeine Theoreme der mathematischen Elastizitätslehre (Integrationstheorie)," pages 55–124; and "Spezielle Ausführungen zur Statik elastischer Körper," in collaboration with A. Timpe, pages 125–214.

ALFONSO DEL RE, professor of descriptive geometry and design in the University of Naples since 1899, died September, 1921. He was born at Calitri, Italy, October 9, 1859. In 1886 he received his doctorate at the University of Naples and was there an assistant in projective geometry, 1886–1889. Among his books are the following: *Lezioni di algebra della logica ad uso degli studenti delle facoltà di matematica e di filosofia e lettere* (Naples, 1907); *Geometria proiettiva ed analitica, Lezioni* (22 + 394 pages, Modena, 1900; 984 pages, Rome, 1900). He was a collaborating editor of *Giornale di Matematiche*, 1910–1913, and in this periodical, as well as in *Rendiconti del Circolo Matematico di Palermo*, and publications of the Modena, Naples and Rome academies, many of his papers are to be found.

DÉSIRÉ ERNEST LEBON, honorary professor at Lycée Charlemagne, Paris, died February 12, 1922. He was born at Audigny, France, August 25, 1846. He was professor of mathematics, or of descriptive geometry, in various lycées

from 1873 to 1898, the year of his appointment as professor of descriptive geometry at Lycée Charlemagne. He is the author of a score of books and many articles. Half of his books deal with descriptive geometry, and other topics in French secondary schools. In America he is probably best known by his *Histoire abrégée de l'Astronomie* (Paris, 1899), by his 7 volumes in *Savants du Jour* series (Poincaré, Picard, Darboux, Appell, Lippmann, Gautier, and Haller, Paris, 1909–1913) and by the first fascicule of his last work, *Table de Caractéristiques de base 30030* ··· (Paris, 1920), reviewed by D. N. Lehmer in an interesting manner for *Bulletin of the American Mathematical Society*, 1921, volume 27, pp. 225–227. He was chief editor, during the 8 years of its existence, of *Bulletin Scientifique*, Paris, 1886–1894, a periodical devoted almost wholly to mathematics.

ALFRED BRAY KEMPE, born in London, England, July 6, 1849, died April 21, 1922. He graduated as twenty-second wrangler at Cambridge in 1872, became a barrister at the Inner Temple in 1873 and bencher in 1909. He was president of the London Mathematical Society 1893–1894. He was elected a fellow of the Royal Society in 1887 (vice-president, 1902), and created a knight in 1921. He published a number of mathematical papers 1872–1895 and his lecture on linkages[1] was reprinted in book form with the title, *How to Draw a Straight Line* (1877). In a paper of 1876 he proved that a linkwork can be found to describe any algebraic curve.[2] Sir Alfred's paper published in the *American Journal of Mathematics*, 1879, was accepted for eleven years as containing a proof (Heawood discovered the flaw) of the famous map-coloring problem[3] mentioned by Möbius in lectures of 1840. In the same *Journal*, for 1913, G. D. Birkhoff made a notable contribution in the discussion of the problem.

HEINRICH SUTER, professor of mathematics in the Gymnasium of Zürich since 1886, died March 17, 1922. He was born at Hedingen, near Zürich, January 4, 1848. After studying at the Universities of Berlin and Zürich he received his doctorate from the latter in 1872. His dissertation was the first edition of the first part of his *Geschichte der mathematischen Wissenschaften*, namely, *Von den ältesten Zeiten bis Ende des XVI. Jahrhunderts*, the second edition of which was published at Zürich in 1873. The first edition of the second part, *Von Anfange des XVII. bis gegen das Ende des XVIII. Jahrhunderts*, appeared in 1875. A Hungarian translation of the first part, by Sandor, was published at Budapest in 1874; and a Russian translation by Manujlow, at Kischinew in 1876. His *Die Mathematik auf den Universitäten des Mittelalters* appeared in a Programm of

[1] For a bibliography of this subject, 1796–1882, see *Bulletin des Sciences Mathématiques et Astronomiques*, vol. 18, 1883, pp. 145–160. Among the articles in this MONTHLY dealing with linkages are the following: "On kinematic geometry—a new inversor" by J. J. Quinn, *1905*, 105–106; "The solution of an equation by a frame" by T. M. Blakslee, *1911*, 159–162; "A cardiograph" by C. M. Hebbert, *1915*, 12–13; "Linkages" by D. H. Leavens, *1915*, 330–334.

[2] The mechanical production of rectilinear motion, aside from its practical importance, has exercised a peculiar fascination on the minds of geometers, and the allied theory of three-bar curves is still being investigated. For an interesting popular account of the theory of linkages see a paper by F. V. Morley in *The Scientific Monthly*, vol. 9, 1919, p. 366. Mr. Morley has an analytic treatment of the problem in *Proceedings of the London Mathematical Society*, June 28, 1922.

[3] See *Encyklopädie der mathematischen Wissenschaften*, III–1–1, 1907, pp. 177–178.

the Kantonschule, Zürich, 1887, and in "Festschrift der Kantonschule zur 39. Versammlung Deutscher Philologen und Schulmänner" published at Zürich in the same year. His most important publications are probably: *Mathematiker-Verzeichnis im Fihrist des Ibn Abî Jakûb an-Nadîm* (Leipzig, 1892), and *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig, 1900–1902, 8 + 278 + 8 + 337 pages) which appeared in volumes 6, 10 and 14 of *Abhandlungen zur Geschichte der Mathematik*. His most recent large work was: *Al-Kwarizmi, Mohammed Ibn Musa. Die astronomischen Tafeln in der Bearbeitung des Maslama Ibn Ahmed Al-Madjzetti und der lateinischen Uebersetzung des Athelhard von Bath auf Grund der Vorarbeiten von A. Björnbo und R. Besthorn. Herausgegeben und kommentiert von H. Suter* (Copenhagen, 1914; 284 pages). He was also an extensive contributor to *Bibliotheca Mathematica* since 1889, and to *Zeitschrift für Mathematik und Physik* since 1884. A tiny portrait of Suter is given on page 17 of Eneström's Bibliotheca Mathematica, General Register, 1887–1896, 1897.

It has been the custom at the United States Military Academy (West Point) and the United States Naval Academy (Annapolis) to have the major part of the instruction committed to officers of the military and naval services, respectively. The study of military tactics has been left to army officers and naval manœuvers as a subject for instruction has never been in the hands of civilians. As a matter of fact, instruction has frequently been delegated to recent graduates of the respective academies, to young men of excellent habits and high ideals but with no pedagogical experience and with neither taste nor opportunity for scholarly research. For many years past the Naval Academy has made an exception of the courses in mathematics which constitute an important share of the midshipman's curriculum, and there has been a large corps of expert civilian instructors in mathematics at Annapolis. Recently a move has been made which threatens to result in the wholesale dismissal of civilians from the teaching staff of the Academy, the future instruction in calculus, mechanics, probability, and other technical mathematical topics to fall to such naval officers as may be spared for this work. It is to be hoped that the future defenders of our shores may not be robbed of the best scientific training available as an outcome of politics or prejudice, whatever decision is made.

Professor ARCHIBALD HENDERSON, of the University of North Carolina, gave a paper on "How the Einstein theory of relativity was verified" before the physics section of the North Carolina Academy of Science at Chapel Hill on May 5 and 6. He was elected president of the Academy for the coming year.

Attention is called to the fact that the date of the Ohio Section meeting should be March 30–31, instead of March 29–30, as announced in the preliminary notice.

Published February 26, 1923.

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RATES OF EXCHANGE.

By HUBERT E. BRAY, The Rice Institute, Houston, Texas.

The general subject of rates of exchange is, of course, a complicated one. In this paper an elementary discussion of the subject is presented, following Cournot,[1] under the special hypothesis that there is no flow of gold between the various markets involved. Thus the rates of exchange are determined by a system of simultaneous linear homogeneous equations in n unknowns, the ratios of which are the rates of exchange between the n markets. The matrix of this system is of a special type which possesses interesting properties.

1. Preliminary Example. For the sake of clearness consider first a special case in which only three financial markets are involved, *e.g.* New York, London, Paris. Suppose that these form a closed system, *i.e.*, that they have dealings only with each other. We will suppose also that transactions are carried on by exchanges of credit, *i.e.*, that there is no flow of gold from one market to another.

Let the markets in question be indicated by subscripts 1, 2, 3, respectively, and suppose that

New York possesses m_{12} pounds of credit at London,
New York possesses m_{13} francs of credit at Paris,
London possesses m_{21} dollars of credit at New York,
etc.

Suppose also that

One dollar at New York is equal in value to C_{12} pounds at London,
One dollar at New York is equal in value to C_{13} francs at Paris,
etc.

The quantities C_{12}, C_{13}, C_{23}, etc., are called *rates of exchange*. Since there is no flow of gold between markets, we must have:

$$\begin{aligned} m_{21} + m_{31} &= m_{12}C_{21} + m_{13}C_{31}, \\ m_{12} + m_{32} &= m_{21}C_{12} + m_{23}C_{32}, \\ m_{13} + m_{23} &= m_{31}C_{13} + m_{32}C_{23}. \end{aligned} \qquad (A)$$

The first equation states that the number of dollars of credit which London and Paris together hold at New York is equal to the combined value in dollars at New York of the number of pounds of credit at London and the number of francs of credit at Paris held by New York. The other equations have a similar meaning.

Since a dollar at New York is worth C_{12} pounds of credit at London, and a

[1] Augustin Cournot, *The Mathematical Principles of the Theory of Wealth*, chapter III, New York, 1897. The equations, which are due to Cournot, were discussed in a course given by Professor Evans at the Rice Institute.